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Analyzing System Reliability Using Fuzzy Weibull Lifetime Distribution

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Abstract Investigation of reliability characteristics under fuzzy environments is proposed in this paper. Fuzzy Weibull distribution and lifetimes of components are using it described. Formulas of a fuzzy reliability function, fuzzy hazard function and their α-cut set are presented. Furthermore, the fuzzy functions of series systems and parallel systems are discussed, respectively. Finally, some numerical examples are presented to illustrate how to calculate the fuzzy reliability characteristics and their α-cut set.

Keywords Fuzzy Reliability Function, Fuzzy Weibull Distribution, Fuzzy Hazard Function, Fuzzy Number.

1 Introduction

The lifetimes are assumed to be random variables. The probability distributions of the random variables (lifetime density function) have crisp parameters. In many situations, the parameters are difficult to determine due to uncertainties and imprecision of data. So it is reasonable to assume the parameters to be fuzzy quantity. In this paper, the lifetimes and repair times of components are assumed to have Weibull distribution with fuzzy parameters. The Weibull distribution function was introduced by Fisher and Tippett in 1928. The Swedish physicist Wallodi Weibull used this probability distribution for describing the lifetime of components with variable failure rate in 1939.

Among the various distributions already studied, the Weibull distribution has been proven to be flexible and versatile at describing monotonic failure rate data. However, for many modern complex systems which exhibit unimodal or bathtub shaped failure rates, the Weibull distribution is inadequate [1]. The most common model used in studies of reliability is Weibull distribution. This distribution is used frequently in different branches of engineering for modeling the fail time. For this purpose, we extend the Weibull distribution in fuzzy environment.

Cai et al. [2-3] gave a different insight by introducing the possibility assumption and fuzzy state assumption to replace the probability and binary state assumptions. Cai *et al.* [4] also discussed the system reliability for coherent system based on the fuzzy state assumption and probability assumption. Cai [5] presented an introduction to system failure engineering and its use of fuzzy methodology. Utkin [6] discussed imprecise reliability models for the general lifetime distribution classes. They both applied the theory of imprecise probability to

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reliability analysis. Guo *et al.* [7] proposed a credibility hazard concept associated with fuzzy lifetimes. Guo *et al.* [8] considered random fuzzy variable modeling on repairable systems. Karpisek *et al.* [9] described two fuzzy reliability models that are based on the Weibull fuzzy distribution. Baloui Jamkhaneh [10] evaluated reliability function using fuzzy exponential lifetime distribution. Also Baloui Jamkhaneh [11] considered reliability estimation under the fuzzy environments. In this paper we propose a general procedure to construct the reliability characteristics and its α-cut set, when the parameters are fuzzy. The parameter of the system is represented by a trapezoidal fuzzy number [12].

The rest of the paper is organized as follows: Section 2 introduces the fuzzy Weibull distribution. Section 3 describes the formalization of fuzzy reliability function. Section 4 describes the formalization of fuzzy hazard function. In suction 5, we discuss the fuzzy reliability function of series and parallel systems. Finally, Section 6 concludes the paper.

2 Fuzzy Weibull distribution

The Weibull distribution is a widely used distribution function in reliability, and its probability density function is as:

$$f(x)=\frac{\beta}{\theta}\left(\frac{x-\delta}{\theta}\right)^{\beta-1}e^{-\left(\frac{x-\delta}{\theta}\right)^{\beta}} \quad , \quad x>\delta,\delta>0,\theta>0,\beta>0, \tag{1}$$

where θ (scale parameter), β (shape parameter) and δ (location parameter) are crisp. The Weibull distribution has the cumulative distribution function:

$$F(t)=1-e^{-\left(\frac{t-\delta}{\theta}\right)^{\beta}} \quad , \quad t>\delta. \tag{2}$$

The shape parameter is what gives the Weibull distribution its flexibility. By changing the value of the shape parameter, this distribution can model a wide variety of lifetime data. If $\beta=1$ the Weibull distribution is identical to the exponential distribution; if $\beta=2$, the Weibull distribution is identical to the Rayleigh distribution; if $\beta=3.25$, the Weibull distribution approximates the normal distribution. The Weibull distribution approximates the lognormal distribution for several values of β. Therefore, the distribution Weibull has great flexibility for modeling different data.

However, sometimes we face situations when the parameters are vague. Therefore, we consider the Weibull distribution with fuzzy parameters. Now consider fuzzy number of $\tilde{\theta}$ replaces θ in Weibull distribution. The distribution function with fuzzy parameter of a random variable is defined as follows:

$$\begin{aligned} F(x,\tilde{\theta})&=\{F(x)[\alpha],\mu_{F(x)}\,|\,F(x)[\alpha]=[F_{\min}(x)[\alpha],F_{\max}(x)[\alpha]],\mu_{F(x)}=\alpha\},\\ F_{\min}(x)[\alpha]&=\inf\{F(x,\theta)\,|\,\theta\in\tilde{\theta}[\alpha]\},\\ F_{\max}(x)[\alpha]&=\sup\{F(x,\theta)\,|\,\theta\in\tilde{\theta}[\alpha]\}. \end{aligned} \tag{3}$$

In this case, the fuzzy probability of event $X \in [c,d]$, $c \geq 0$ ($\tilde{P}(c \leq X \leq d)$) and its $\alpha-$cut set computes as follows: [13].

$$\tilde{P}(c \leq X \leq d)[\alpha] = \{\int_c^d \frac{\beta}{\theta}(\frac{x-\delta}{\theta})^{\beta-1} e^{-(\frac{x-\delta}{\theta})^{\beta}} dx \,\Big|\, \theta \in \tilde{\theta}[\alpha]\} = [P^L[\alpha], P^U[\alpha]], \tag{4}$$

for all α, where

$$\begin{aligned} P^L[\alpha] &= \min\{\int_c^d \frac{\beta}{\theta}(\frac{x-\delta}{\theta})^{\beta-1} e^{-(\frac{x-\delta}{\theta})^{\beta}} dx \,\Big|\, \theta \in \tilde{\theta}[\alpha]\}, \\ P^U[\alpha] &= \max\{\int_c^d \frac{\beta}{\theta}(\frac{x-\delta}{\theta})^{\beta-1} e^{-(\frac{x-\delta}{\theta})^{\beta}} dx \,\Big|\, \theta \in \tilde{\theta}[\alpha]\}. \end{aligned} \tag{5}$$

3 Fuzzy reliability functions

Fuzzy reliability is based on the fuzzy set theory of Zadeh. Fuzzy reliability (fuzzy survival) function ($\tilde{S}(t)$) is the fuzzy probability in which a unit survives beyond time t. Let the random variable X denote lifetime of a system components, also let X has density function $f(x,\tilde{\theta})$ and fuzzy cumulative distribution function $\tilde{F}_X(t) = \tilde{P}(X \leq t)$ where parameter $\tilde{\theta}$ is a fuzzy number, and then the fuzzy reliability function at time t is defined as:

$$\tilde{S}(t) = \tilde{P}(X > t) = 1 - \tilde{F}_X(t) = \{[1-F_{\max}(x)[\alpha], 1-F_{\min}(x)[\alpha]], \mu_{F(x)} = \alpha\}, \ t > 0, \tag{6}$$

and the fuzzy unreliability function $\tilde{Q}(t)$ is the fuzzy probability of failure or the fuzzy probability of an item failing in the time interval $[0,t]$

$$\tilde{Q}(t) = \tilde{P}(X \leq t) = \tilde{F}_X(t) \ , \ t > 0. \tag{7}$$

Suppose that we want to calculate the reliability of a component, such that the lifetime random variable has fuzzy Weibull distribution. So we represent parameter $\tilde{\theta}$ with a trapezoidal fuzzy number as $\tilde{\theta} = (a_1, a_2, a_3, a_4)$ such that we can describe a membership function $\xi_{\tilde{\theta}}(x)$ in the following manner:

$$\xi_{\tilde{\theta}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & , \quad a_1 \leq x < a_2 \\ 1 & , \quad a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} & , \quad a_3 < x \leq a_4 \end{cases} \tag{8}$$

The $\alpha-$cut $\tilde{\theta}$ is denoted as follows:

$$\tilde{\theta}[\alpha]=[a_1+(a_2-a_1)\alpha, a_4-(a_4-a_3)\alpha]. \tag{9}$$

So fuzzy reliability function of a component is as follows:

$$\tilde{S}(t)[\alpha]=\{\int_t^{\infty}\frac{\beta}{\theta}(\frac{x-\delta}{\theta})^{\beta-1}e^{-(\frac{x-\delta}{\theta})^{\beta}}dx \,\Big|\, \theta\in\tilde{\theta}[\alpha]\}=\{e^{-(\frac{t-\delta}{\theta})^{\beta}} \,\Big|\, \theta\in\tilde{\theta}[\alpha]\}. \tag{10}$$

According to that the $e^{-(\frac{t-\delta}{\theta})^{\beta}}$ increasing θ, then the α-cuts of fuzzy reliability function is as:

$$\tilde{S}(t)[\alpha]=[\exp\{-(\frac{t-\delta}{a_1+(a_2-a_1)\alpha})^{\beta}\},\exp\{-(\frac{t-\delta}{a_4-(a_4-a_3)\alpha})^{\beta}\}], \tag{11}$$

$\tilde{S}(t)[\alpha]$ is a two dimensional function in terms of α and t ($0\le\alpha\le 1$ and $t>0$). For t_0, this is a trapezoidal fuzzy number and membership function of $\tilde{S}(t_0)$ is as follows:

$$\xi_{\tilde{S}(t_0)}(x)=\begin{cases}\dfrac{x-\exp\{-(\frac{t_0-\delta}{a_1})^{\beta}\}}{\exp\{-(\frac{t_0-\delta}{a_2})^{\beta}\}-\exp\{-(\frac{t_0-\delta}{a_1})^{\beta}\}} & , \exp\{-(\frac{t_0-\delta}{a_1})^{\beta}\}\le x<\exp\{-(\frac{t_0-\delta}{a_2})^{\beta}\}\\ 1 & , \exp\{-(\frac{t_0-\delta}{a_2})^{\beta}\}\le x\le\exp\{-(\frac{t_0-\delta}{a_3})^{\beta}\}\\ \dfrac{\exp\{-(\frac{t_0-\delta}{a_4})^{\beta}\}-x}{\exp\{-(\frac{t_0-\delta}{a_4})^{\beta}\}-\exp\{-(\frac{t_0-\delta}{a_3})^{\beta}\}} & , \exp\{-(\frac{t_0-\delta}{a_3})^{\beta}\}<x\le\exp\{-(\frac{t_0-\delta}{a_4})^{\beta}\}\end{cases} \tag{12}$$

In this method, for any fixed α and arbitrary t reliability curve is like a band whose width depends on the ambiguity parameter of θ. The lesser uncertainty value results in less bandwidth, and if the parameter gets a crisp value, the lower and upper bounds will become equal, which means that reliability curve is in a classic state. This reliability band has properties as follows:

(i) $\tilde{S}(0)[\alpha]=\tilde{1}$, i.e. no one starts off dead,
(ii) $\tilde{S}(\infty)[\alpha]=\tilde{0}$, i.e. everyone dies eventually,
(iii) $\tilde{S}(t_1)[\alpha]\tilde{\ge}\tilde{S}(t_2)[\alpha]\Leftrightarrow t_1\le t_2$, i.e. band of $\tilde{S}(t)[\alpha]$ declines monotonically,
($\tilde{S}(t_1)[\alpha]\tilde{\ge}\tilde{S}(t_2)[\alpha]$ if and only if $S^L(t_1)[\alpha]\ge S^L(t_2)[\alpha]$ and $S^U(t_1)[\alpha]\ge S^U(t_2)[\alpha]$ for all $\alpha\in[0,1]$, where "$\tilde{\ge}$" means "fuzzy greater than or equal to"),
(iv) for any fixed α and $\beta\le 1$ reliability band have convex functions.
Fuzzy mean time to failure (FMTTF) is the expected time to failure. According definition of

Buckley [13] FMTTF of this fuzzy system is a fuzzy number and can be calculated as follows:

$$M\tilde{T}TF[\alpha] = \{\int_0^\infty xf\ (x)dx\ \left|\theta \in \tilde{\theta}[\alpha]\right.\} = \{\int_0^\infty S\ (t)dt\ \left|\theta \in \tilde{\theta}[\alpha]\right.\} = [P^L[\alpha], P^U[\alpha]].$$
$$P^L[\alpha] = \min\{\int_0^\infty S\ (t)dt\ \left|\theta \in \tilde{\theta}[\alpha]\right.\}\ ,\ \ P^U[\alpha] = \max\{\int_0^\infty S\ (t)dt\ \left|\theta \in \tilde{\theta}[\alpha]\right.\}. \quad (13)$$

When the lifetime random variable has fuzzy Weibull distributed then:

$$\begin{aligned} M\tilde{T}TF[\alpha] &= \{\theta\,\Gamma(1+\beta^{-1})\left|\theta \in \tilde{\theta}[\alpha]\right.\} \\ &= [(a_1 + (a_2 - a_1)\alpha)\Gamma(1+\beta^{-1}), (a_4 - (a_4 - a_3)\alpha)\Gamma(1+\beta^{-1})]. \end{aligned} \quad (14)$$

Accordingly (14), the following membership function is obtained:

$$\xi_{M\tilde{T}TF}(x) = \begin{cases} \dfrac{x - a_1\Gamma(1+\beta^{-1})}{(a_2 - a_1)\Gamma(1+\beta^{-1})} & , \quad a_1\Gamma(1+\beta^{-1}) \le x < a_2\Gamma(1+\beta^{-1}) \\ 1 & , \quad a_2\Gamma(1+\beta^{-1}) \le x \le a_3\Gamma(1+\beta^{-1}) \\ \dfrac{a_4\Gamma(1+\beta^{-1}) - x}{(a_4 - a_3)\Gamma(1+\beta^{-1})} & , \quad a_3\Gamma(1+\beta^{-1}) < x \le a_4\Gamma(1+\beta^{-1}) \end{cases} \quad (15)$$

4 Fuzzy hazard functions

In fuzzy reliability theory, the fuzzy hazard function plays an attractive role. We will propose the concept of a fuzzy hazard function based on the fuzzy probability measure and call its $\alpha-$cut hazard band. The fuzzy hazard function $\tilde{h}(t)$ is the fuzzy conditional probability of an item failing in the interval t to (t + dt) given that it has not failed by time t. Hazard function is also known as the instantaneous failure rate. Mathematically, we would define the fuzzy hazard function as

$$\begin{aligned} \tilde{h}(t)[\alpha] &= \lim_{\Delta t \to 0} \frac{\tilde{P}(t < X < t + \Delta t\left|X > t\right.)}{\Delta t} = \{\lim_{\Delta t \to 0} \frac{S\,(t) - S\,(t + \Delta t)}{\Delta t S\,(t)}\left|\theta \in \tilde{\theta}[\alpha]\right.\} \\ &= \{\frac{-S^{'}(t)}{S\,(t)}\left|\theta \in \tilde{\theta}[\alpha]\right.\} = \{\frac{f\ (t)}{S\,(t)}\left|\theta \in \tilde{\theta}[\alpha]\right.\}. \end{aligned} \quad (16)$$

The fuzzy Weibull distributed with $\delta = 0$ has fuzzy hazard function is as follows

$$\tilde{h}(t)[\alpha] == \{\frac{\beta}{\theta}(\frac{t}{\theta})^{\beta-1}\left|\theta \in \tilde{\theta}[\alpha]\right.\}, \quad (17)$$

$\tilde{h}(t)[\alpha]$ is a two dimensional function in terms of α and t ($0 \le \alpha \le 1$ and $t > 0$). In this method, for every $\alpha-$cut, hazard curve is like a band. One of the most important aspects of the Weibull distribution is performance of its hazard band. This band with change the parameters of density function can decrease, increase and be constant. For $\beta = 1$, hazard function is a fuzzy number constant for every t, whereas $\beta > 1$, leads to an increasing band, and hence can be considered to model wear-out, as often deemed appropriate for mechanical units, and $\beta < 1$ leads to decreasing band, hence modeling wear-in of a product as often advocated for electronic units (see Figure 2 and Figure 3). An increasing hazard band at time t indicates fuzzy failure probability of component in time $(t, t+\Delta)$ is more than fuzzy failure probability the previous period the same length, that is, components wear during the time.

Example. Let lifetime of the component is modeled by a Weibull distribution with fuzzy parameter $\tilde{\theta}$ and $\delta = 0$ that $\tilde{\theta} = (1.5, 1.52, 1.54, 1.56)$. Then $\alpha-$cut of fuzzy reliability function, fuzzy hazard function and FMTTF are given by
(i) Fuzzy reliability band

$$\tilde{\theta}[\alpha] = (1.5 + 0.02\alpha, 1.56 - 0.02\alpha),$$

$$\tilde{S}(t)[\alpha] = [\exp\{-(\frac{t}{1.5 + 0.02\alpha})^{\beta}\}, \exp\{-(\frac{t}{1.56 - 0.02\alpha})^{\beta}\}], \tag{18}$$

for all α.

(1) If $t = 0.5$ then fuzzy reliability is as:

$$\xi_{\tilde{S}(0.5)}(x) = \begin{cases} \dfrac{x - \exp\{-(0.333)^{\beta}\}}{\exp\{-(0.329)^{\beta}\} - \exp\{-(0.33)^{\beta}\}}, & \exp\{-(0.33)^{\beta}\} \le x < \exp\{-(0.329)^{\beta}\} \\ 1, & \exp\{-(0.329)^{\beta}\} \le x \le \exp\{-(0.325)^{\beta}\} \\ \dfrac{\exp\{-(0.321)^{\beta}\} - x}{\exp\{-(0.321)^{\beta}\} - \exp\{-(0.325)^{\beta}\}}, & \exp\{-(0.325)^{\beta}\} < x \le \exp\{-(0.321)^{\beta}\} \end{cases} \tag{19}$$

(2) If $\alpha = 0$ then reliability band is as:

$$\tilde{S}(t)[0] = [\exp\{-(0.67t)^{\beta}\}, \exp\{-(0.64t)^{\beta}\}]. \tag{20}$$

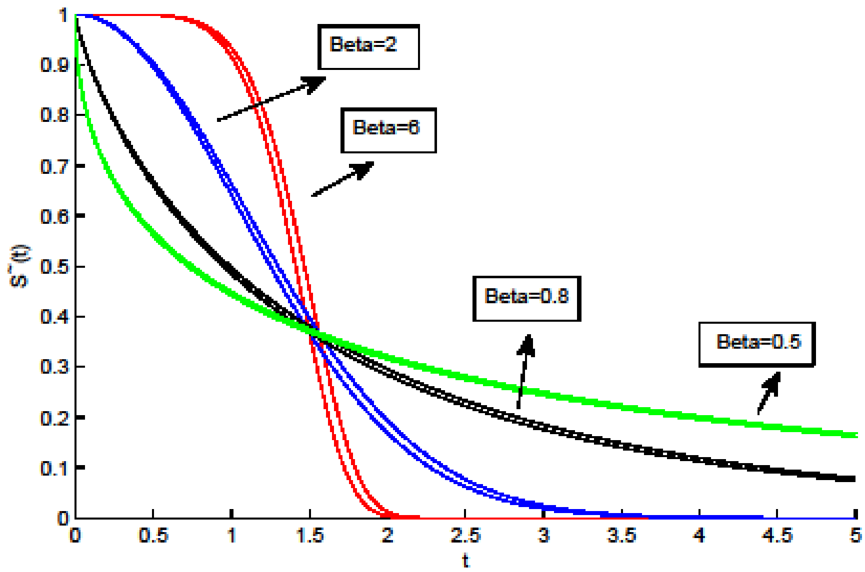

Fig .1 α – cut of fuzzy survival band ($\alpha = 0$)

(ii) Fuzzy hazard band:

$$\tilde{h}(t)[\alpha]=[\frac{\beta}{1.56-0.02\alpha}(\frac{t}{1.56-0.02\alpha})^{\beta-1},\frac{\beta}{1.5+0.02\alpha}(\frac{t}{1.5+0.02\alpha})^{\beta-1}]. \quad (21)$$

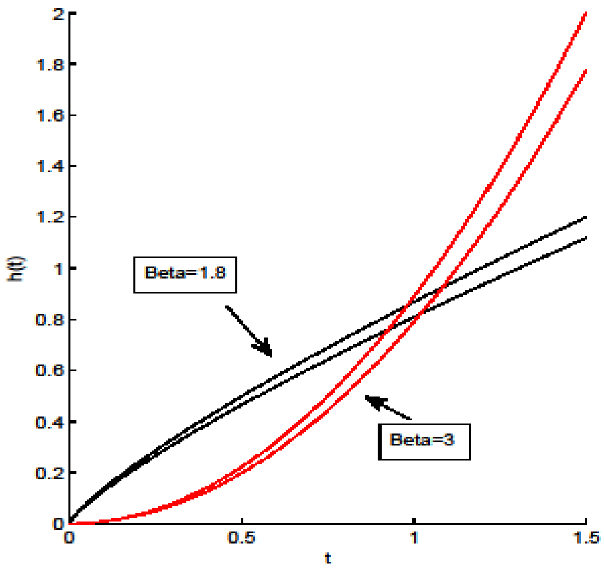

Fig .2 α – cut of fuzzy hazard band ($\alpha = 0$)

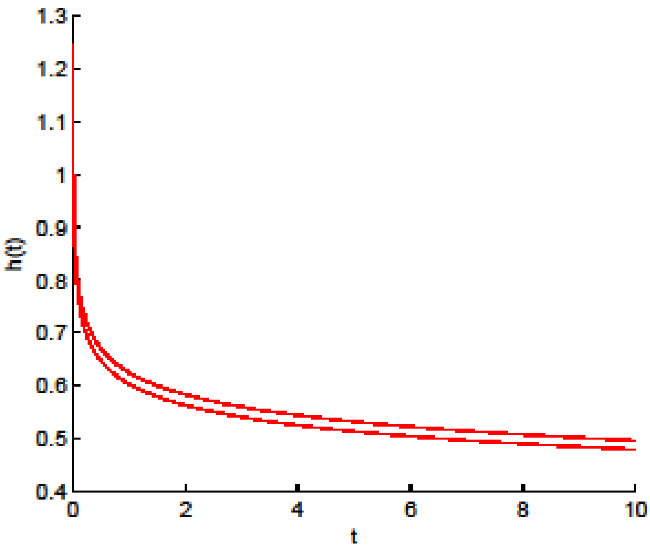

Fig .3 $\alpha-$cut of fuzzy hazard band ($\alpha=0, \beta=0.9$)

Figure 2 and Figure 3 show the behavior of fuzzy hazard band in various conditions.

(iii) FMTTF

$$M\tilde{T}TF[\alpha](\beta)=\{\theta\,\Gamma(1+\beta^{-1})\big|\theta\in\tilde{\theta}[\alpha]\}$$
$$=[(1.5+0.02\alpha)\Gamma(1+\beta^{-1}),(1.56-0.02\alpha)\Gamma(1+\beta^{-1})]. \quad (22)$$

According to the behavior of the gamma function, the value of FMTTF at $\beta=2.166$ has a minimum and minimum value of FMTTF is as follows:

$$M\tilde{T}TF[\alpha](2.166)=[1.3284+0.0177\alpha,1.3815-0.0177\alpha].$$

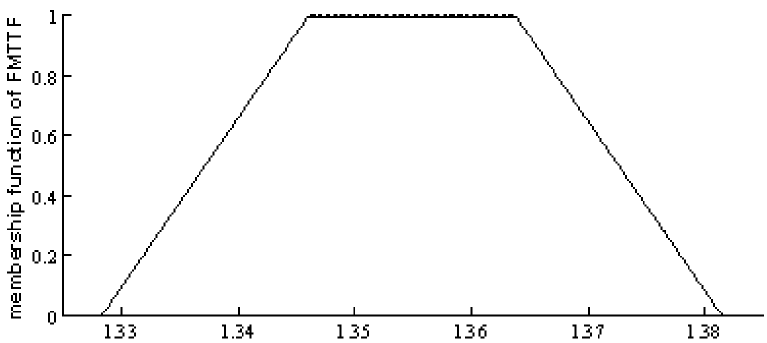

Fig .4 membership function of FMTTF

5 Series and parallel system

In the case series systems of m identical and independent components, $\alpha-$cut set of fuzzy reliability function $\tilde{R}(t)[\alpha]$ with modeled fuzzy Weibull distribution is as:

$$\tilde{R}(t)[\alpha]=\{S(t)^m \,|\theta\in\tilde{\theta}[\alpha]\}=[e^{-\frac{m(t-\delta)^\beta}{(a_1+(a_2-a_1)\alpha)^\beta}},\ e^{-\frac{m(t-\delta)^\beta}{(a_4-(a_4-a_3)\alpha)^\beta}}]. \tag{23}$$

In case parallel systems of m identical and independent components, its $\alpha-$cut set of fuzzy reliability function with modeled fuzzy Weibull distribution is as:

$$\tilde{R}(t)[\alpha]=\{1-Q^m(t)\,|\theta\in\tilde{\theta}[\alpha]\}=[1-(1-e^{-\frac{(t-\delta)^\beta}{(a_1+(a_2-a_1)\alpha)^\beta}})^m,\ 1-(1-e^{-\frac{(t-\delta)^\beta}{(a_4-(a_4-a_3)\alpha)^\beta}})^m]. \tag{24}$$

6 Conclusions

The fuzzy reliability function and fuzzy hazard function have been successfully investigated in this paper. Whenever the lifetimes of components and parameters contain randomness and fuzziness respectively, the approach of reliability theory based on traditional statistical analysis may be inappropriate. Fuzzy system reliability is based on the concept of fuzzy set and fuzzy probability theory in our method. In this paper, the scale parameter was considered as fuzzy trapezoidal number. In further research, the shape and location parameters can be considered fuzzy separately or combined. Also in the further research is required to investigate some important topics in fuzzy reliability theory such as mean residual life.

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