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Analyses of a Markovian queue with two heterogeneous servers and working vacation

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Abstract This paper analyzes an M/M/2 queueing system with two heterogeneous servers. Both servers goes on vacation when there is no customers waiting for service after this server 1 is always available but the other goes on vacation whenever server 2 is idle. The vacationing server however, returns to serve at a low rate as an arrival finds the other server busy. The system is analyzed in the steady state using matrix geometric method.

Keywords: Working Vacation (WV), Matrix Geometric Solution.

1 Introduction

We analyze an M/M/2 queue with working vacations (WVs), in which the server works with variable service rates rather than completely stops service during his/her vacation period. Such a vacation is called a working vacation. Each server starts a vacation when the system is empty at his service completion epoch. If server 1 returns from vacation and find server 2 busy, and there is no customers in the system; then, he stays idle and ready for serving new arrivals. If the server 2 returns from a WV to find the system not empty, he immediately switches to the original service rate.

The queueing systems with server vacations or WVs have been investigated by many researchers. Past work may be divided into two categories: (i) the case of server vacation and (ii) the case of WV. In the case of server vacation, the readers are referred to the survey paper by Doshi [1] and monograph of Takagi [2]. The works of Takagi [2] and Doshi [3] focus on a single server. As for multiple server system with vacations Zhang and Tian [4,5] gave a plenty analysis of M/M/c with synchronous multiple/single vacations of partial servers. In the case of working vacation, Servi and Finn [6] first examined an M/M/1 queue with multiple WVs where inter-arrival times, service times during service period, and vacation times are all exponentially distributed. They developed the explicit formulae for the mean and variance number of customers in the system, and the mean and variance waiting time in the system. Later Wu and Takagi [7] extended Servi and Finn's [6] discuss the model M/M/1/WV queue to an M/G/1/WV queue. In [8] the model of [6] is generalized to the M/G/1queue.

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In queueing models vacations can be classified as single and multiple server involving single vacation and multiple vacations. The server may take a vacation at a random time, after serving at most K customers or after all the customers in the queue are served. The queueing systems with single or multiple vacation have been introduced by Levy and Yachiali [9]. The literature about vacation models is growing rapidly which includes survey papers by Teghem [10], Doshi [1,3] and the monograph by Takagi [2]. We can find general models in Tian and Zhang [11]. In 2007, Li and Tian [12] first introduced vacation interruption policy and studied an M/M/1 queue. Recently, [13] have driven a new elegant explicit solution for a two heterogeneous servers queue with impatient behavior. In [14] the author considers an M/M/R queue with vacations, in which the server works with different service rates rather than completely terminates service during his vacation period. In [14] obtained an M/M/1 retrial queue with WVs, vacation interruption, Bernoulli feedback and N-policy simultaneously. During the WV period, customers can be served at a lower rate. Using the matrix-analytic method, we get the necessary and sufficient condition for the system to be stable.

The study on multi-server queueing systems generally assumes the servers to be homogeneous in which the individual service rates are the same for all the servers in the system. This assumption may be valid only when the service process is mechanically or electronically controlled. In a queueing system with human servers, the above assumption can hardly realized. It is common to observe server rendering service to identical jobs at different service rates. This reality leads to modeling such multi-server queueing systems with heterogeneous servers, that is the service time distributions may be different for different servers. Levy and Yechiali [15] have discussed the vacation policy in a multi-server Markovian queue. They have considered a model with 's' homogeneous servers and exponentially distributed vacation times. Using partial generating function technique, the system size has been obtained. Kao and Narayanan [16] have discussed the M/M/s queue with multiple vacations of the servers using a matrix geometric approach. Gray et al [17] have discussed a single counter queueing model involving multiple servers with multiple vacations. A researcher have discussed an M/M/s queue with multiple vacation and 1-limited service. Neuts and Lucantoni [18] have analyzed the M/M/s queueing systems where the servers are subject to random breakdowns and repairs. Baba [19] extended Servi and Finn's [6] M/M/1/WV queue to a GI/M/1/WV queue. They not only assumed general independent arrival, they also assumed service times during service period, service times during vacation period as well as vacation times following exponential distribution. Furthermore, Baba [19] derived the steady- state system length distributions at arrival and arbitrary epochs.

Neuts and Takahashi [20] observed that for queueing systems with more than two heterogeneous servers analytical results are intractable and only algorithmic approach could be used to study the steady state behavior of the system. Krishna Kumar and Pavai Madheswari [21] analyzed M/M/2 queueing system with heterogeneous servers where the servers go on vacation in the absence of customers waiting for service. Based on this observation, Krishnamoorthy and Sreenivasan [22] analyzed M/M/2 queueing system with heterogeneous servers where one server remains idle but the other goes on vacation in the absence of waiting customers. In this paper we discuss an M/M/2 queueing system with heterogeneous servers where both servers go on vacation in the absence of customers, for the remaining times server 1 is always in the system and server 2 go for vacation whenever it is idle.

2 Quasi Birth- and-death process Model

Consider an M/M/2 queueing system with two heterogeneous servers. Arrivals of customers follow a Poisson process with parameter λ. Let μ_1 and μ_2 be the service rates of server1 and server2 respectively, where $\mu_1 \neq \mu_2$. The duration of vacation periods are assumed to be independent and identically distributed exponential random variables with parameters θ_1 and θ_2. During the vacation, if an arrival finds server 1 is busy then server 2 returns to serve the customer at a lower rate. To be precise server 2 serve this customers at the rate $\theta\mu_2, 0 < \theta \leq 1$. At this vacation gets over server 2 instantaneously switches over to the normal service rate μ_2, upon completion of a service at low rate if no customer is waiting for service then go for vacation and if at least one customer is waiting for service then server 2 is busy with normal service rate. The arriving customers are served under the first-come-first-served (FCFS) discipline. The vacation queueing model with heterogeneous server under consideration can be formulated as a continuous time Markov chain (CTMC). The possible states of the system at any epoch are represented by (i, j) where $i \geq 0$ denotes the number of customers in the system and j=0,1,2,3 denotes the status of the servers. The state (0,0) represents there is no customers in the system and both servers are on vacation; after the state (0,0) server 1 is always available in the system the state $(i,1)$ represent ($i \geq 0$) customers are in the system and server 1 is busy in the system while server 2 is on vacation; the state $(i,2)$ represent $i \geq 0$ customers in the system and server 1 is busy and server 2 is in working vacation mode; the state $(i,3)$ represent $i \geq 0$ customers in the system and both servers are busy in the system with normal mode.

Let Q denotes the infinitesimal generator of the continuous time Markov chain (CTMC) corresponding to this Q and is in the format of a quasi-birth-and –death (QBD) process. Define the levels 0,1,2,…, as the set of the states 0={(0,0)}, $1 = \{(1,0),(1,1),(1,2)\}$, and $i = \{(i,0),(i,1),(i,2),(i,3)\}$ if $i \geq 2$. The state transition diagram of the system is as follows

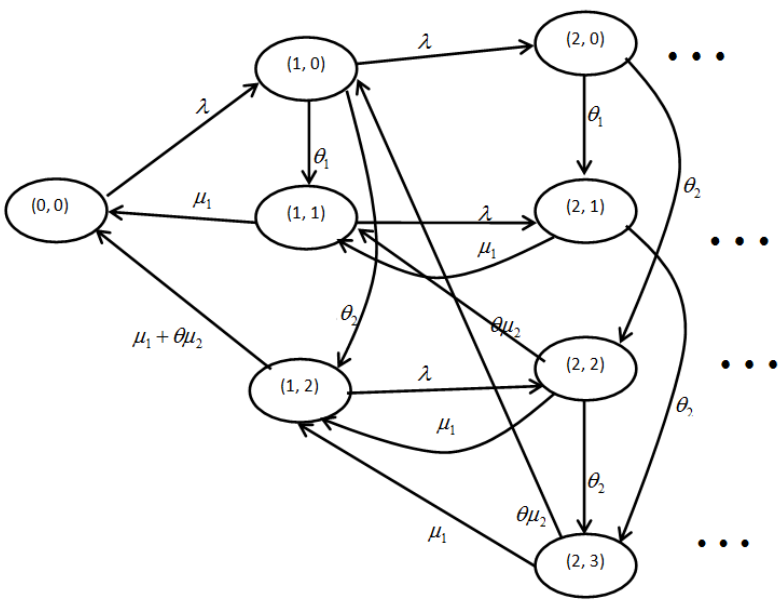

Fig. 1

Using the lexicographical sequence for the states the infinite generator can be written as

$$Q=\begin{bmatrix} B_{00} & B_{01} & & & & & & \\ B_{10} & B_{11} & B_{12} & & & & & \\ & B_{21} & A_1 & A_0 & & & & \\ & & A_2 & A_1 & A_0 & & & \\ & & & A_2 & A_1 & A_0 & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

The sub-matrices A_0, A_1 and A_2 are of order 4x4 and are given by

$$A_0=\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad A_2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_1+\theta\mu_2 & 0 \\ 0 & 0 & 0 & \mu_1+\mu_2 \end{bmatrix}$$

$$A_1=\begin{bmatrix} -(\lambda+\theta_1+\theta_2) & \theta_1 & 0 & \theta_2 \\ 0 & -(\lambda+\mu_1+\theta_2) & 0 & \theta_2 \\ 0 & 0 & -(\lambda+\mu_1+\theta\mu_2+\theta_2) & \theta_2 \\ 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2) \end{bmatrix}$$

The boundary matrices are defined by

$$B_{00}=-\lambda, B_{01}=(\lambda,0,0), \quad B_{10}=\left[0,\mu_1,(\mu_1+\theta\mu_2)\right]^T$$

$$B_{11}=\begin{bmatrix} -(\lambda+\theta_1+\theta_2) & \theta_1 & \theta_2 \\ 0 & -(\lambda+\mu_1) & 0 \\ 0 & 0 & -(\lambda+\mu_1+\theta\mu_2) \end{bmatrix}$$

$$B_{12}=\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{bmatrix}, \quad B_{21}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & \mu_1 & \theta\mu_2 \\ 0 & \mu_2 & \mu_1 \end{bmatrix}$$

Define the matrix $A=A_0+A_1+A_2$. Then A is 4x4 matrix and it can be written as

$$A=\begin{bmatrix} -(\theta_1+\theta_2) & \theta_1 & \theta_2 & 0 \\ 0 & -\theta_2 & 0 & \theta_2 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.1 Rate Matrix

To analyze this QBD process, a very important matrix in evaluating the performance measures is the matrix R. it is known as the rate matrix of the Markov chain Q and it has the minimal non-negative solution of the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0 \qquad (2.1)$$

Since the matrices A_0, A1, and A_2 are of order 4x4 upper triangular, R is also a 4x4 upper triangular matrix

Lemma 2.1

(a) The quadratic equation

$\mu_1 z^2 - (\lambda + \mu_1 + \theta_2)z + \lambda = 0$ includes the service rate μ_1 of server 1 and the vacation parameter θ_2 of server 2.

Therefore, the above quadratic equation has two different real roots $r_1 < r_1^*$ and $0 < r_1 < 1,\ r_1^* > 1$

where

$$r_1 = \frac{1}{2\mu_1}\left((\lambda + \mu_1 + \theta_2) - \sqrt{(\lambda + \mu_1 + \theta_2)^2 - 4\lambda\mu_1}\right) \text{ and}$$

$$r_1^* = \frac{1}{2\mu_1}\left((\lambda + \mu_1 + \theta_2) + \sqrt{(\lambda + \mu_1 + \theta_2)^2 - 4\lambda\mu_1}\right)$$

(b) The quadratic equation

$(\mu_1 + \theta\mu_2)z^2 - (\lambda + \mu_1 + \theta\mu_2 + \theta_2)z + \lambda = 0$ includes the service rate μ_1 of server 1 and the working vacation parameter $\theta\mu_2$ of server 2.

Therefore, the above quadratic equation has two different real roots $r_2 < r_2^*$ and $0 < r_2 < 1,\ r_2^* > 1$

where,

$$r_2 = \frac{1}{2(\mu_1 + \theta\mu_2)}\left((\lambda + \mu_1 + \theta\mu_2 + \theta_2) - \sqrt{(\lambda + \mu_1 + \theta\mu_2 + \theta_2)^2 - 4\lambda(\mu_1 + \theta\mu_2)}\right)$$

and

$$r_2^* = \frac{1}{2(\mu_1 + \theta\mu_2)}\left((\lambda + \mu_1 + \theta\mu_2 + \theta_2) + \sqrt{(\lambda + \mu_1 + \theta\mu_2 + \theta_2)^2 - 4\lambda(\mu_1 + \theta\mu_2)}\right)$$

(c) The quadratic equation

$(\mu_1 + \mu_2)z^2 - (\lambda + \mu_1 + \mu_2)z + \lambda = 0$ includes the two service rates μ_1 and μ_2 of server 1 and server 2 respectively, but it does not include the parameters θ_1 and θ_2. From this it is clear that the above quadratic is for the case when both two servers are busy in service and it has two different real roots $r_3 = \rho = \lambda(\mu_1 + \mu_2)^{-1}$ and $r_3^* = 1$.

From the matrix R, we find that the spectral radius $sp(R) = \max(r_{0,0}, r_{1,1}, r_{2,2}, r_{3,3}) < 1$ if and only if $\rho < 1$. Hence, we can prove that $\rho < 1$ is the necessary and sufficient condition for that the stability of the process $\{L(t), J(t), t \geq 0\}$ is to be positive recurrent (see Neuts [18]).

Theorem 2.1

If $\rho < 1$, the matrix equation (2.1) has the minimal non-negative solution as follows:

$$R=\begin{bmatrix} r_0 & r_{0,1} & 0 & r_{0,3} \\ 0 & r_1 & 0 & r_{1,3} \\ 0 & 0 & r_2 & r_{2,3} \\ 0 & 0 & 0 & \rho \end{bmatrix}$$

Where $r_{0,0}=\dfrac{\lambda}{\lambda+\theta_1+\theta_2}$, $r_{0,1}=\dfrac{\theta_1 r_{0,0}}{\mu_1(r_{1,1}^{*}-r_{0,0})}$, $r_{0,2}=\dfrac{\theta_2 r_{0,0}}{\theta\mu_2(r_{2,2}^{*}-r_{0,0})}$,

$$r_{0,3}=\frac{\left[\left(1+\dfrac{a_{1,1}}{(1-a_{1,1})}\right)\dfrac{\theta_1\theta_2 a_{0,0}}{\mu_1(a_{1,1}^{*}-a_{0,0})}\right]+\theta_2 a_{0,0}}{(\mu_1+\mu_2)(1-a_{0,0})}, \qquad r_{1,3}=\frac{r_{1,1}\theta_2}{(\mu_1+\mu_2)(1-r_{1,1})},$$

$$r_{2,3}=\frac{\theta_2 a_{2,2}}{(\mu_1+\mu_2)(1-a_{2,2})}.$$

Proof.

Since the coefficient matrices of equation (1) are all upper triangular, so let

$$R=\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & 0 & a_{3,3} \end{bmatrix} \tag{2.2}$$

$$R^2A_2=\begin{bmatrix} 0 & \mu_1\sum_{i=0}^{1}a_{0,i}a_{i,1} & (\mu_1+\theta\mu_2)\sum_{i=0}^{2}a_{0,i}a_{i,2} & (\mu_1+\mu_2)\sum_{i=0}^{3}a_{0,i}a_{i,3} \\ 0 & \mu_1 a_{1,1}^2 & (\mu_1+\theta\mu_2)\sum_{i=1}^{2}a_{1,i}a_{i,2} & (\mu_1+\mu_2)\sum_{i=1}^{3}a_{1,i}a_{i,3} \\ 0 & 0 & (\mu_1+\theta\mu_2)a_{2,2}^2 & (\mu_1+\mu_2)\sum_{i=2}^{3}a_{2,i}a_{i,3} \\ 0 & 0 & 0 & (\mu_1+\mu_2)a_{3,3}^2 \end{bmatrix} \tag{2.3}$$

$$RA_1=\begin{bmatrix} -(\lambda+\theta_1+\theta_2)a_{0,0} & a_{0,0}\theta_1+-(\lambda+\mu_1+\theta_2)a_{0,1} & -(\lambda+\mu_1+\theta\mu_2+\theta_2)a_{0,2} & \theta_2\sum_{i=0}^{2}a_{0,i}-(\lambda+\mu_2+\mu_1)a_{0,3} \\ 0 & -(\lambda+\mu_1+\theta_2)a_{1,1} & -(\lambda+\mu_1+\theta\mu_2+\theta_2)a_{1,2} & \theta_2\sum_{i=1}^{2}a_{1,i}-(\lambda+\mu_2+\mu_1)a_{1,3} \\ 0 & 0 & -(\lambda+\mu_1+\theta\mu_2+\theta_2)a_{2,2} & \theta_2 a_{2,2}-(\lambda+\mu_2+\mu_1)a_{2,3} \\ 0 & 0 & 0 & -(\lambda+\mu_2+\mu_1)a_{3,3} \end{bmatrix} \tag{2.4}$$

Substituting (3),(4) and A_0 into (1) gives the following set of equations

$$-(\lambda+\theta_1+\theta_2)a_{0,0}+\lambda=0 \tag{2.5}$$

$$\mu_1 a_{1,1}^2-(\lambda+\mu_1+\theta_2)a_{1,1}+\lambda=0 \tag{2.6}$$

$$(\mu_1+\theta\mu_2)a_{2,2}^2-(\lambda+\theta\mu_2+\mu_1+\theta_2)a_{2,2}+\lambda=0 \tag{2.7}$$

$$(\mu_1+\mu_2)a_{3,3}^2-(\lambda+\mu_1+\mu_2)a_{3,3}+\lambda=0 \tag{2.8}$$

$$\mu_1 \sum_{i=0}^{1} a_{0,i} a_{i,1} + a_{0,0}\theta_1 - (\lambda + \mu_1 + \theta_2) a_{0,1} = 0 \tag{2.9}$$

$$\left(\mu_1 + \theta\mu_2\right) \sum_{i=0}^{2} a_{0,i} a_{i,2} + a_{0,0}\theta_2 - (\lambda + \theta\mu_2 + \mu_1 + \theta_2) a_{0,2} = 0 \tag{2.10}$$

$$\left(\mu_1 + \theta\mu_2\right) \sum_{i=1}^{2} a_{1,i} a_{i,2} - (\lambda + \theta\mu_2 + \mu_1 + \theta_2) a_{1,2} = 0 \tag{2.11}$$

$$(\mu_1 + \mu_2) \sum_{i=0}^{3} a_{0,i} a_{i,3} + \theta_2 \sum_{i=0}^{2} a_{0,i} - (\lambda + \mu_1 + \mu_2) a_{0,3} = 0 \tag{2.12}$$

$$(\mu_1 + \mu_2) \sum_{i=1}^{3} a_{1,i} a_{i,3} + \theta_2 \sum_{i=1}^{2} a_{1,i} - (\lambda + \mu_1 + \mu_2) a_{1,3} = 0 \tag{2.13}$$

$$(\mu_1 + \mu_2) \sum_{i=2}^{3} a_{2,i} a_{i,3} + \theta_2 a_{2,2} - (\lambda + \mu_1 + \mu_2) a_{2,3} = 0 \tag{2.14}$$

From equation (2.5), we get

$$a_{0,0} = \frac{\lambda}{\lambda + \theta_1 + \theta_2}, \tag{2.15}$$

Solving equation (2.9), we get

$$a_{0,1} = \frac{\theta_1 a_{0,0}}{\mu_1 (a_{1,1}^* - a_{0,0})}, \tag{2.16}$$

Solving equation (2.11), we get

$$a_{1,2} = 0 \tag{2.17}$$

Using (2.17) in equation (2.10), we get

$$a_{0,2} = 0 \tag{2.18}$$

Using (2.16) in equation (2.12), we get

$$a_{0,3} = \frac{\left[\left(1 + \frac{a_{1,1}}{(1 - a_{1,1})}\right) \frac{\theta_1 \theta_2 a_{0,0}}{\mu_1 (a_{1,1}^* - a_{0,0})}\right] + \theta_2 a_{0,0}}{(\mu_1 + \mu_2)(1 - a_{0,0})}, \tag{2.19}$$

Using (2.17) in equation (2.13), we get

$$a_{1,3} = \frac{\theta_2 a_{1,1}}{(\mu_1 + \mu_2)(1 - a_{1,1})},$$

Finally from equation (2.14), we have

$$a_{2,3} = \frac{\theta_2 a_{2,2}}{(\mu_1 + \mu_2)(1 - a_{2,2})},$$

It is clear that the above equations have unique non-negative solution. Therefore, this non-negative solution must be the minimal.

3 Stationary Distribution

Let L and J be the stationary random variables for the queue length and the status of the servers. Denote the stationary probability by

$$x_{ij} = P\{L = i, J = j\}$$
$$= \lim_{t\to\infty} P\{L(t) = i, J(t) = j\}, \ (i, j) \in \Omega.$$

Under the stability condition $\rho < 1$, the stationary probability vector x of the generator Q exists. This stationary probability vector x is partitioned as $x = (x_0, x_1, x_2, ...)$ where x_0 is a scalar. $x_1 = (x_{10}, x_{11}, x_{12})$ and $x_i = (x_{i0}, x_{i1}, x_{i2}, x_{i3})$ for $i \geq 2$

Based on the matrix geometric solution method in (see Neuts [18]), the stationary probability vector x is given by

$$x_0 B_{00} + x_1 B_{10} = 0, \tag{3.1}$$
$$x_0 B_{01} + x_1 B_{11} + x_2 B_{21} = 0, \tag{3.2}$$
$$x_1 B_{12} + x_2 \left(A_1 + RA_2\right) = 0, \tag{3.3}$$
$$x_i = x_2 R^{i-2}, \ i = 3, 4, 5, ... \tag{3.4}$$

and the normalizing equation

$$x_0 + x_1 e_1 + x_2 (I - R)^{-1} e_2 = 1 \tag{3.5}$$

Where I is a 4x4 identity matrix, e_1 is a 3x1 column vector and e_2 is a 4x1 column vector with all their elements equal to one.

$$A_1 + RA_2 = \begin{bmatrix} -(\lambda + \theta_1 + \theta_2) & \left(1 + \frac{r_0}{r_1^* - r_0}\right)\theta_1 & \theta_2 & \left(\frac{1}{1 - r_0}\right)\left[\left(1 + \frac{r_1}{1 - r_1}\right)\left(\frac{\theta_1 \theta_2 r_0}{\mu_1 (r_1^* - r_0)}\right) + \theta_2 r_0\right] \\ 0 & -h_1 & \theta_2 & \left(\frac{r_1 \theta_2}{1 - r_1}\right) \\ 0 & 0 & -h_2 & \left(1 + \frac{r_2}{1 - r_2}\right)\theta_2 \\ 0 & 0 & 0 & -(\mu_1 + \mu_2) \end{bmatrix}$$

The equations (3.1), (3.2) and (3.3) can be written as the set of equations;

$$-\lambda x_0 + \mu_1 x_{11} + (\mu_1 + \theta\mu_2) x_{12} = 0 \tag{3.6}$$
$$\lambda x_0 - (\lambda + \theta_1 + \theta_2) x_{10} = 0 \tag{3.7}$$
$$\theta_1 x_{10} - \left(\lambda + \mu_1\right) x_{11} + \mu_1 x_{21} + \mu_1 x_{22} + \mu_2 x_{23} = 0 \tag{3.8}$$
$$\theta_2 x_{10} - \left(\lambda + \theta\mu_2 + \mu_1\right) x_{12} + \theta\mu_2 x_{22} + \mu_1 x_{23} = 0 \tag{3.9}$$
$$\lambda x_{10} - (\lambda + \theta_1 + \theta_2) x_{20} = 0 \tag{3.10}$$
$$\lambda x_{11} + \left(1 + \frac{r_0}{(r_1^* - r_0)}\right) x_{20} - h_1 x_{21} = 0 \tag{3.11}$$

$$\lambda x_{12}+\theta_2 x_{20}+\theta_2 x_{21}-h_2 x_{22}=0 \tag{3.12}$$

$$\phi x_{20}+\left(\frac{r_1\theta_2}{(1-r_1)}\right)x_{21}+\left(1+\frac{r_2}{1-r_2}\right)\theta_2 x_{22}-(\mu_1+\mu_2)x_{23}=0 \tag{3.13}$$

Where, $\phi=\left(\frac{1}{1-r_0}\right)\left[\left(1+\frac{r_1}{1-r_1}\right)\left(\frac{\theta_1\theta_2 r_0}{\mu_1(r_1^*-r_0)}\right)+\theta_2 r_0\right]$

Solving equation (3.7), we get

$x_{10}=r_0x_0$

Solving equation (3.10), we get

$x_{20}=r_0^2x_0$

Solving the remaining equations, we have

$x_{11}=\left[\frac{\lambda-(\mu_1+\theta\mu_2)\beta_1}{\mu_1}\right]x_0$,

$x_{12}=\beta_1x_0$

$x_{21}=\beta_2x_0$

$x_{22}=\beta_3x_0$

$x_{23}=\beta_4x_0$

Where, $\beta_1=\left[\frac{\phi_4\phi_6+\phi_5h_1\mu_1}{\lambda\phi_6(\mu_1+\theta\mu_2)-(\lambda\phi_3+\phi_2h_2)(\mu_1h_1)}\right]$

$$\beta_2=\left[\frac{\phi_4-\lambda\beta_1(\mu_1+\theta\mu_2)}{(\mu_1h_1)}\right]$$

$$\beta_3=\left[\frac{\lambda\beta_1+\theta_2r_0^2+\theta_2\beta_2}{h_2}\right]$$

$$\beta_4=\left[\frac{\phi r_0^2+\left(\frac{r_1\theta_2}{1-r_1}\right)\beta_2+\left(1+\frac{r_2\theta_2}{1-r_2}\right)\beta_3}{(\mu_1+\mu_2)}\right]$$

$$\phi_1=\left[(\mu_1+\mu_2)\theta_2r_0+\mu_1\phi r_0^2\right]$$

$$\phi_2=-\left[(\mu_1+\mu_2)(\lambda+\mu_1+\theta\mu_2)\right]$$

$$\phi_3=\left[(\mu_1+\mu_2)\theta\mu_2+\left(1+\frac{r_2}{1-r_2}\right)\theta_2\mu_1\right]$$

$$\phi_4=\left[\lambda^2+\left(1+\frac{r_0}{r_1^*-r_0}\right)\theta_1\mu_1r_0^2\right]$$

$$\phi_5=\left[\phi_1h_2+\theta_2\phi_3r_0^2\right]$$

$$\phi_6=\left[\theta_2\phi_3+\left(\frac{r_1\theta_2\mu_1h_2}{1-r_1}\right)\right]$$

$$h_1 = \frac{1}{2}\left[(\lambda+\mu_1+\theta_2)+\sqrt{(\lambda+\mu_1+\theta_2)^2-4\lambda\mu_1}\right]$$

$$h_2 = \frac{1}{2}\left[(\lambda+\mu_1+\theta\mu_2)+\sqrt{(\lambda+\mu_1+\theta\mu_2+\theta_2)^2-4\lambda(\mu_1+\theta\mu_2)}\right]$$

$$(I-R)^{-1} = \begin{bmatrix} \left(\frac{1}{1-r_0}\right) & \left(\frac{r_{0,1}}{(1-r_0)(1-r_1)}\right) & 0 & \left(\frac{\left[r_{1,3}r_{0,1}+r_{0,3}(1-r_1)\right]}{(1-r_0)(1-r_1)(1-\rho)}\right) \\ 0 & \left(\frac{1}{1-r_1}\right) & 0 & \left(\frac{r_{1,3}}{(1-r_1)(1-\rho)}\right) \\ 0 & 0 & \left(\frac{1}{1-r_2}\right) & \left(\frac{r_{2,3}}{(1-r_2)(1-\rho)}\right) \\ 0 & 0 & 0 & \left(\frac{1}{1-\rho}\right) \end{bmatrix} \quad (3.14)$$

Substituting (3.14) and I where, I be the unit matrix of order 4x4 in (3.5), we get

$$y = \begin{bmatrix} 1+r_0+\left\{\frac{\lambda-(\mu_1+\theta\mu_2)\beta_1}{\mu_1}\right\}+\beta_1+\left(\frac{1}{(1-r_{3,3})}\right)\beta_4 \\ +\left\{\frac{1}{(1-r_{0,0})}+\frac{r_{0,1}}{(1-r_{0,0})(1-r_{1,1})}+\frac{r_{1,3}r_{0,1}+r_{0,3}(1-r_{1,1})}{(1-r_{0,0})(1-r_{1,1})(1-r_{3,3})}\right\}r_0^2 \\ +\left\{\frac{1}{(1-r_{1,1})}+\frac{r_{1,3}}{(1-r_{1,1})(1-r_{3,3})}\right\}\beta_2+\left\{\frac{1}{(1-r_{2,2})}+\frac{r_{2,3}}{(1-r_{2,2})(1-r_{3,3})}\right\}\beta_3 \end{bmatrix}$$

$$x_0 = y^{-1}$$

Let L denote the stationary queue length at an arbitrary epoch. Therefore, the mean, second moment and variance of the number of customers in the system can be obtained as

$$E(L) = x_1e_1 + 2x_2e_2 + x_2R\left[(I-R)^{-2}+2(I-R)^{-1}\right]e_2$$

$$E(L^2) = x_1e_1 + 4x_2e_2 + x_2\left[(I-R)^{-1}+(I-R)^{-2}+2(I-R)^{-3}-4I\right]e_2$$

and

$$\text{var}(L) = E(L^2) - \left[E(L)\right]^2$$

4 The Busy Period

For the working vacation model with heterogeneous servers, the busy period is defined to be the interval between the arrival of a customer to an empty system and first epoch thereafter when the system becomes empty again. Thus, the busy period is the first passage time from state (1,0) to state (0,0). For the working vacation model, busy cycle for the system is the time interval between two successive departures, which leave the system empty. Therefore, the busy cycle is the first return time to state (0,0) with at least one visit to any other state.

To discuss busy period analysis, the notion of fundamental period (14) should be briefly reviewed. For the QBD process described above, starting from a state in level i, where $i \geq 3$, the first passage time to a state in level i-1 constitutes a fundamental period. The cases i=2, i=1, and i=0 corresponding to the boundary states need to be discussed separately. Because of the structure of the QBD process, the first passage time distribution is invariant in i.

Let $G_{jj'}(k,t)$ denote the conditional probability that a QBD process starting in the state (i,j) at time t=0, the first visit to level i-1 occurs no later than time t, into the state (i-1,j') and exactly k transitions occur to the left during the first passage time.

The matrix representation of joint transforms of $G(z,s)$ is defined by

$$G(z,s) = \sum_{k=1}^{\infty} z^k \int_0^{\infty} e^{-st} dG(k,t) \text{ for } |Z| \leq 1, \text{ Re } s \geq 0 \tag{4.1}$$

$G(z,s)$ satisfies the relation $G^n(z,s) = \left[G^1(z,s)\right]^n$ for $n \geq 1$ (4.2)

and define $G^1(z,s) = G(z,s)$

then the matrix $G(z,s)$ satisfies the equation

$$G(z,s) = z\left(sI - A_1\right)^{-1} A_2 + \left(sI - A_1\right)^{-1} A_0 G^2(z,s) \tag{4.3}$$

To discuss the boundary conditions for the vacation model, for i = 1,2, let $G_{jj'}^{(i,i-1)}(k,t)$ denote the conditional probability that a QBD process, starting in the state (i, j) at time 0, reaches the level i-1 for the first time no later than t, after exactly k transitions to the left and does so by entering the state (i-1, j'). let $G^{(i,i-1)}(z,s)$ denote the transform matrix corresponding to $G^{(i,i-1)}(k,t)$. Using the similar argument in (4.3), we get

$$G^{(2,1)}(z,s) = z\left(sI - A_1\right)^{-1} B_{21} + \left(sI - A_1\right)^{-1} A_0 G(z,s) G^{(2,1)}(z,s), \tag{4.4}$$

$$G^{(1,0)}(z,s) = z\left(sI - B_{11}\right)^{-1} B_{10} + \left(sI - B_{11}\right)^{-1} B_{12} G^{(2,1)}(z,s) G^{(1,0)}(z,s), \tag{4.5}$$

$$G^{(0,0)}(z,s) = \left[\frac{\lambda}{s+\lambda}, 0, 0\right] G^{(1,0)}(z,s), \tag{4.6}$$

Where $G^{(0,0)}(z,s)$ is the joint transform of the recurrence time to state (0,0) with at least one visit to a state other than state (0,0). Note that $G^{(1,0)}(z,s)$ is of 3x1. The Laplace Stieltjes transform (LST) for the length of a busy period is then given by the first element of $G^{(1,0)}(1+,s)$, that is, the element corresponds to the first passage from state (1,0) to the state (0,0). The busy cycle comprises an idle period and a busy period. The LST for the length of a busy cycle is given by $G^{(0,0)}(1+,s)$.

Let the matrices

$$G = \lim_{\substack{z \to 1- \\ s \to 0+}} G(z,s), \quad G_{2,1} = \lim_{\substack{z \to 1- \\ s \to 0+}} G^{(2,1)}(z,s), \quad G_{1,0} = \lim_{\substack{z \to 1- \\ s \to 0+}} G^{(1,0)}(z,s), \text{and}$$

$$G_{0,0} = \lim_{\substack{z \to 1- \\ s \to 0+}} G^{(0,0)}(z,s). \tag{4.7}$$

The positive recurrence of the process Q implies that G, $G_{2,1}$, $G_{1,0}$ and $G_{0,0}$ are all stochastic and $G_{0,0} = (1,1,1)^T$. Let $C_0 = (-A_1)^{-1} A_2$, $C_2 = (-A_1)^{-1} A_0$. in (14) it has been proved that G is the minimal non negative solution to the equation $G = C_0 + C_2 G^2$.

Taking the $\lim\limits_{\substack{z\to 1- \\ s\to 0+}}$ on both sides of the equations (4.4), (4.5), (4.6) and using (4.7), we get

$$G_{2,1} = -(A_1 + A_0 G)^{-1} B_{21,} \tag{4.8}$$

$$G_{1,0} = -(B_{11} + B_{12} G_{21})^{-1} B_{10,} \tag{4.9}$$

$$G_{0,0} = [1\ 0\ 0] G_{10}. \tag{4.10}$$

If we found the value of the matrix R then matrix G can be computed using the following result (15),

$$G = -(A_1 + RA_2)^{-1} A_2. \tag{4.11}$$

Let $M_1 = \left. \dfrac{\partial G(z,s)}{\partial s} \right|_{z=1,s=0}$ it has been proved (14) that the matrix M_1 can be computed by successive substitutions in

$$M_1 = -A_1^{-1} G + C_2 \left(G M_1 + M_1 G \right),$$

With 0 as the starting value for M_1. For the boundary states, differentiating (4.4), (4.5) and (4.6) with respect to s and setting s=0, z=1, we get

$$M_{2,1} = -(A_1 + A_0 G)^{-1} (I + A_0 M) G_{2,1},$$

$$M_{1,0} = -(B_{11} + B_{12} G_{21})^{-1} (I + B_{12} M_{21}) G_{1,0},$$

It is clear that $M_{1,0}$ can be computed recursively starting with M_1 in decreasing order. Assuming $M_{0,0}$ as the mean recurrence time for state (0,0), a similar operation leads to

$$M_{0,0} = \left[\frac{1}{\lambda}\ 0\ 0 \right] G_{1,0} + [1\ 0\ 0] M_{1,0}.$$

Where

$$M_{2,1} = \left. \frac{\partial G^{(2,1)}(z,s)}{\partial s} \right|_{z=1,s=0}$$

$$M_{1,0} = \left. \frac{\partial G^{(1,0)}(z,s)}{\partial s} \right|_{z=1,s=0}$$

$$M_{0,0} = \left. \frac{\partial G^{(0,0)}(z,s)}{\partial s} \right|_{z=1,s=0}$$

It is note that the mean length of a busy cycle is $M_{0,0}$. The first element of the column vector $M_{1,0}$ yields the mean length of a busy period, E(B). the mean length of a server vacation period, E(v) is $\dfrac{1}{\theta_1 + \theta_2}$. The mean length of a server busy period is obtained as E(B)-E(V).

Table 1

λ	$\theta=0.7,\theta_1=3,\theta_2=1$					
	$\mu_1=10,\mu_2=5$			$\mu_1=8,\mu_2=8$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	Var(L)
2	0.8055	1.7748	1.1260	0.8572	1.9081	1.1733
4	1.7254	5.7183	2.7414	1.8597	6.3006	2.8422
6	2.8105	12.7750	4.8762	3.0508	14.5064	5.1993
8	4.1633	25.3177	7.9847	4.4998	29.4524	9.2040
10	6.0124	49.9098	13.7611	6.3654	57.1470	16.6291
12	9.3096	115.8571	29.1889	9.1843	114.6471	30.2961

Table 2

λ	$\theta=0.7,\theta_1=3,\theta_2=1$			$\theta=0.7,\theta_1=1,\theta_2=3$		
	$\mu_1=5,\mu_2=10$			$\mu_1=10,\mu_2=5$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	Var(L)
2	1.0451	2.4825	1.3901	0.8797	1.9192	1.1452
4	2.3418	9.0528	3.5686	1.6970	5.2289	2.3490
6	3.8745	22.8721	7.8607	2.4972	10.1618	3.9259
8	5.7453	49.2668	16.2579	3.4201	17.9936	6.2963
10	8.2651	98.1197	29.8078	4.7140	33.0041	10.7822
12	12.5174	206.9532	50.2688	7.2500	77.5638	25.0016

λ	$\theta=0.7,\theta_1=1,\theta_2=3$			$\theta=0.7,\theta_1=1,\theta_2=3$		
	$\mu_1=8,\mu_2=8$			$\mu_1=5,\mu_2=10$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	Var(L)
2	0.8914	1.9297	1.1350	0.9752	2.1334	1.1824
4	1.7044	5.1609	2.2559	1.8498	5.6507	2.2290
6	2.4718	9.7623	3.6524	2.6646	10.6711	3.5712
8	3.3112	16.6156	5.6518	3.5903	18.6125	5.7220
10	4.3985	28.3871	9.0399	4.9158	34.1461	9.9812
12	6.2137	55.7571	17.1471	7.5645	80.8240	23.6019

Table 3

λ	$\theta=0.7,\theta_1=2,\theta_2=2$					
	$\mu_1=10,\mu_2=5$			$\mu_1=8,\mu_2=8$		
	$E[L]$	$E[L^2]$	Var(L)	$E[L]$	$E[L^2]$	Var(L)
2	0.8393	1.8341	1.1296	0.8685	1.8947	1.1404
4	1.6930	5.3562	2.4901	1.7434	5.4858	2.4463
6	2.5817	10.9024	4.2373	2.6285	10.9963	4.0875
8	3.6175	19.8615	6.7753	3.6109	19.4918	6.4533
10	5.0411	36.8464	11.4337	4.8623	34.0356	10.3935
12	7.7514	85.6532	25.5685	6.8842	66.4325	19.0400

Table 4

	$\theta = 0.7, \theta_1 = 2, \theta_2 = 2$ $\mu_1 = 5, \mu_2 = 10$		
	$E[L]$	$E[L^2]$	Var(L)
2	0.9908	2.2200	1.2383
4	1.9797	6.4819	2.5628
6	2.9588	13.1594	4.4048
8	4.0781	24.0372	7.4065
10	5.6392	44.5591	12.7591
12	8.6188	101.0972	26.8131

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