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A heuristic method to calculate the real internal rate of return (RIRR)

H. Jafari*, M. Faraji, P. Khaleghi Ardehali

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Abstract The Internal Rate of Return (IRR) is used in economics as a popular method for project evaluation. Its major challenge, however, is to calculate all real rates. This paper presents an algorithm that can be used to calculate all return rates through introducing a heuristic approach based on several mathematical theorems. In order to better understand the different steps of the proposed algorithm, an overall description of the subject was first provided; it was then illustrated with more details through providing some examples. For the purpose of validation and accuracy of the results, the future value of the rates was also calculated at the end of each example. The results of the research showed that the new method had a relatively high efficiency to calculate all real rates. In addition, in the approach proposed in this paper, there is no need for the use of differentiation, and even recurring return rates are also identifiable.

Keyword: Capital Investment Analysis, Cost of Capital, Internal Rate of Return, Present Value.

1 Introduction

Econometrics deals with the systematic study of economic phenomena using observed data. In other words, econometrics is the science of statistical analysis of economic models, so that statistics, economic theory, and mathematics are the foundations of econometrics [1].

Although many econometric methods imply the application of statistical models, econometrics can be differentiated from other branches of statistics using some specific economic data indicators. Economic data are mainly based on observation and are not obtained from controlled experiments. Since economic units interact with each other, the observed data indicate a complex economic balance, rather than a relational simple behavior due to precedence or technology. Hence, econometrics will provide methods for identifying and estimating models with several unknowns. These methods allow the researcher to provide a cause-effect inference in conditions other than controlled testing conditions [2].

* **Corresponding Author. (✉)**
E-mail: Hossein_jafari_123@yahoo.com (H. Jafari)

H. Jafari
Young Researchers and Elite Club, Arak Branch, Islamic Azad Univercity, Arak, Iran.

M. Faraji
Department of Industrial Engineering, Islamic Azad University of Arak, Arak, Iran.

P. Khaleghi Ardehali
Department of Industrial Engineering, Karaj Branch, Islamic Azad University, Karaj, Iran.

In general, giving empirical content to economic relations to test economic theories, as well as predicting, deciding, and assessing a policy or decision can be considered as econometric goals [3].

Engineering economics is one of the branches of economics that discusses analytical methods used to estimate costs and determine the value of systems, products, and services. It includes a set of techniques that simplifies the process of comparing selectable options based on economic principles. The engineering economics is actually a tool for choosing the best or, in other words, the most cost-effective option among the options available to engineers [4].

The need for engineering economics was felt when engineers turned to economic analysis of decisions on engineering projects. The engineering economics is, in fact, the heart of the decision-making process. The decisions in the engineering economics include a number of basic elements such as cash flows, time, and interest rates. Finally, the engineering economics helps us make better and more economical decisions using logical and mathematical methods [5].

It is clear to anyone that over time, the value of money does not remain constant and is constantly changing. This has led to the emergence of equilibrium methods, as well as the internal return rate and different other methods in the economic assessments. All the said methods are based on algebraic and mathematical rules. As far as the role of mathematics in economics is concerned, it should be said that in order for economics to be regarded as an academic approach, one must resort to mathematics [6].

Among the most important methods for evaluating projects are Internal Rate of Return (IRR), Present Value Method (PV) and Future Value Method (FV), which managers and investors among those methods have considered the method of internal return more.

2 Problem statement

The results of evaluating the projects using all engineering approaches are the same, but the method of internal return rate is more common compared to the other methods. Because in this method, the criteria for accepting or rejecting a project are based on the IRR criterion, and most managers and decision makers have a better understanding of this method. In fact, the balance between incomes and expenses is called the internal return rate. The IRR method facilitates decision-making through making a simple comparison of the IRR and the Minimum Attractive Rate of Return (MARR) possible [7].

This is despite the fact that the IRR method also has problems. One of these problems is how to calculate the internal return rates.

Graphical methods are often used to calculate return rates, one of the weaknesses of this approach is that the repeating rates and the accuracy of the rates are uncertain.

In this paper, an approach is introduced for calculating all real IRRs using a heuristic method and some mathematical theorems. Therefore, the present value method, the future value method, and the method of internal return rate are first reviewed, and then the theorems required to present the new approach are proven. It should be noted that there is no need to use differentiation in the approach proposed in this article. After proving the theorems and introducing a new method, a flowchart is drawn up which includes all steps of the proposed algorithm, at the end some numerical examples are proposed and solved by the proposed method.

3 Background of the research problem

Here are some of the most important studies in this area:
Vukčević and Vujadinović introduced the equilibrium techniques and internal rate of return in 2008 [3].

In 1988, Lehman proposed a re-investment plan to address the multi-rate problem of projects and economic plans. Based on the recommendations of Lehman, a single-rate situation can be achieved through the use of some earnings on investments and transfer them to the future flows [7].

Moradi *et al.* (2012) examined the multi-rate problems in uncertain conditions (fuzzy) and showed that the sensitivity of projects and uncertain plans can be analyzed using the Hazen's approach and fuzzy calculations [8].

In 2012, Eils introduced some computational errors in the present value method and introduced a more precise relationship to calculate the present value of a cash flow with a given cost of capital [9].

Russell and Richard (1982) presented a method that allowed multi-rate projects based on the formation of several tracks. These tracks are extractable based on the original cash flow coefficients and the Pascal's triangle [10].

In a study, Jafari and Sheykhan (2015) reviewed multi-rate projects. They presented a new method that is parallel to the Hazen's method and has a simpler and more concise process for proving the theorems [11].

4 Internal rate of return method

A cash flow is a limited or an unlimited sequence of monetary values in the form of $X = (x_0, x_1, \ldots)$. The amount of money received at the beginning of the financial period is equal to x_0 and the amount received after the *t* period is equal to x_t . The time horizon T, as well as $X = (x_0, x_1, \ldots, x_T)$, are considered for a limited cash flow. When the interest rate is equal to r, the present value of the cash flow X is obtained as follows:

$$PV(X|r) = \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \tag{1}$$

$IRR(X)$is a set of interest rates for which $PV(X|r) = 0$. These interest rates are called zeros or cash flow return rate X. If there is only one return rate and that rate is higher than the market interest rate, the investment in cash flow X is profitable; otherwise the investment in cash flow X will not be profitable [3].

The following mathematical model can be used to find the return rates:

$$Min\ f = |PV(X|r)| \tag{2}$$

$s.t.$

$$PV(X|r) = \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \tag{3}$$

$r \in \mathbb{R}$

5 Introduction of the proposed approach

In the proposed method, several theorems are first needed, which are explained separately in the following. Then, the method used by the present paper is fully described with the help of these theorems.

Theorem (1): Suppose that $P(z) = a_n z^n + \cdots + a_1 z^1 + a_0$ is a complex polynomial, then all the zeros of P are put in the following disk, *i.e.*$\{z: |z| < 1 + a\}$, so that $a = \max_{0 \le j \le n} \left|\frac{a_j}{a_n}\right|$ [12].

Theorem (2): If $X = (x_0, x_1, \ldots, x_T)$is an arbitrary cash flow and *r* is the fixed cost of capital, then all real internal return rates will be placed in the interval $[-b-2, +b]$, where the value of b is calculated as follows:

$$b = \max_{0 \le t \le T} \left|\frac{x_t}{x_T}\right| \tag{4}$$

Proof:
According to the definition of the present value method, cash flow X is calculated through the following equation:

$$PV(X|r) = \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \tag{5}$$

And in order to find the return rates, it is enough to consider the Equation (1) as equal to zero and calculate the roots:

$$PV(X|r) = \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} = 0 \tag{6}$$

Now the two sides of the previous equation are multiplied in $(1+r)^T$, so we have:

$$FV(X|r) = \sum_{t=0}^{T} x_t (1+r)^{T-t} = 0 \tag{7}$$

Now, if it is assumed that $\begin{cases} z = 1 + r \\ b_T = x_{T-t} \;_{;t=0,1,\ldots,T} \end{cases}$, then equation(7) can be rewritten as follows:

$$\sum_{t=0}^{T} b_t z^n = b_T z^T + b_{T-1} z^{T-1} + \cdots b_2 z^2 + b_1 z^1 + b_0 = 0 \tag{8}$$

Now, according to Theorem (1), it is easy to say:

$$\begin{cases} |z| < 1 + b \\ b = \max_{0 \le t \le T} \left|\frac{b_t}{b_T}\right| \end{cases} \tag{9}$$

In the continuation of the proofs of the theorem, by returning the variables to the initial state, one can write:

$$\begin{cases} |1 + r| < 1 + b \\ b = \max_{0 \le t \le T} \left|\frac{x_t}{x_0}\right| \end{cases} \tag{10}$$

By simplifying Equation (10), we have:

$$\begin{cases} |1 + r| < 1 + b \\ b = \max_{0 \le t \le T} \left|\frac{x_t}{x_0}\right| \end{cases} \Rightarrow \begin{cases} -b - 2 < r < b \\ b = \max_{0 \le t \le T} \left|\frac{x_t}{x_0}\right| \end{cases} \tag{11}$$

In the previous equation, *r* is of a mixed (imaginary) nature, and its roots are in a disk centered at -1 with a radius of *b*+1. But if we want to find the real roots (real *r*), we simply

delete the imaginary part of this area, so it can be said that the considered rates are in the range of $[-1-(b+1), -1+(b+1)] = [-b-2\,, +b]$.
The proof is complete.

Theorem (3): Suppose that $P(z) = a_n z^n + \cdots + a_1 z^1 + a_0$ is a polynomial with real coefficients, then P can have a maximum number of n real roots.

Proof by contradiction:

We assume that P has N roots in the form of $z_1, z_2, \ldots, z_n, \ldots, z_N$, so the P statement can be written as$P(z) = \gamma \prod_{j=1}^{N}(z - z_j)\,, \gamma \in \mathbb{R} - \{0\}$.

Through multiplication and rewriting the operation of $P(z) = \gamma \prod_{j=1}^{N}(z - z_j)$, it is determined that P is a polynomial with a degree greater than n, and this result is in contradiction with the assumption of the theorem. Therefore, the proof of contradiction is nullified and the rule is still valid.
The proof is complete.

Theorem (4): If $\mathrm{X} = (x_0, x_1\,, \ldots, x_T)$ is an arbitrary cash flow and r is the fixed capital cost, then X can have a maximum T return rate.

Proof:
According to the definition of the internal rate of return, the roots of the next equation should be calculated in order to calculate the return rates X:

$$PV(X|r) = \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} = 0 \tag{12}$$

By multiplying the two sides of Equation (12) in $(1+r)^T$, we have:

$$x_0(1+r)^T + x_1(1+r)^{T-1} + \cdots + x_{T-1}(1+r)^1 + x_T = 0 \tag{13}$$

It is clear that Equation (13) is a real polynomial of degree T, and according to Theorem (3) it cannot have more than T roots (return rate).
The proof is complete.

Lemma (1): If P is a polynomial of $P(z) = \sum_{j=1}^{n} a_j\, z^j$ and the function f is defined as $f(z) = |P(z)|$, then the roots of P and f are the same.

Proof of the first part:
Suppose that z_0 is a root for P, then:

$$P(z_0) = 0 \Rightarrow |P(z_0)| = 0 \Rightarrow f(z_0) = 0 \tag{14}$$

Therefore, z_0 is also the root of f.
Proof of the second part:
Suppose that z_0 is a root for f, then:

$$f(z_0) = 0 \Rightarrow |P(z_0)| = 0 \Rightarrow P(z_0) = 0 \tag{15}$$

Finally, it can be said that the proof is complete.

6 Recommended method for finding real return rates

Suppose that all the roots of the polynomial $P(z) = \sum_{j=1}^{n} a_j\, z^j$ are in the interval $[\alpha_1, \beta_1]$. Now if we divide the said interval into n equal parts, then the value of the function $f(z) =$

$|P(z)|$ is calculated in the partition points. Naturally, some of the values of f are closer to zero, and the possibility of the existence of roots around these points, which we call them z_1^1, is more than the other points. Therefore, if a new boundary is defined as $\left[\alpha_2 = z_1^1 - \frac{\beta_1-\alpha_1}{n}, \beta_2 = z_1^1 + \frac{\beta_1-\alpha_1}{n}\right]$, this time, a smaller region can be searched to find the roots with a greater accuracy; and the point of discovery at this stage is called z_1^2. By continuing this process until a certain repetition, for example, k, a root can be obtained with an arbitrary approximation such as z_1^k (note that in each step to reduce the search space, the equation $\left[\alpha_i = z_1^{i-1} - \frac{\beta_{i-1}-\alpha_{i-1}}{n}, \beta_i = z_1^{i-1} + \frac{\beta_{i-1}-\alpha_{i-1}}{n}\right]$ is used. After determining the value of each root z_j^k, P and f are updated through decomposing polynomials. Now, based on the Theorem (3), this algorithm should be stopped after n times of execution, since the objective function can maximally have n roots. In the following, in order to find the internal rates of return, we act like before, with the exception that we make the initial settings as follows:

$$\begin{cases} f(r) = FV(X|r) \\ [\alpha_1 = -b-2\,, \beta_1 = b] \\ b = \max_{0\le t\le T}\left|\frac{x_t}{x_0}\right| \\ n = T \end{cases} \tag{16}$$

Note: When partitioning, it is better to choose the number of n or T larger than 20 in order to set aside only 90% of the previous search space in each new iteration and help to accurately and quickly find return rates.

In Figure 1, the flowchart contains the steps of the proposed algorithm.

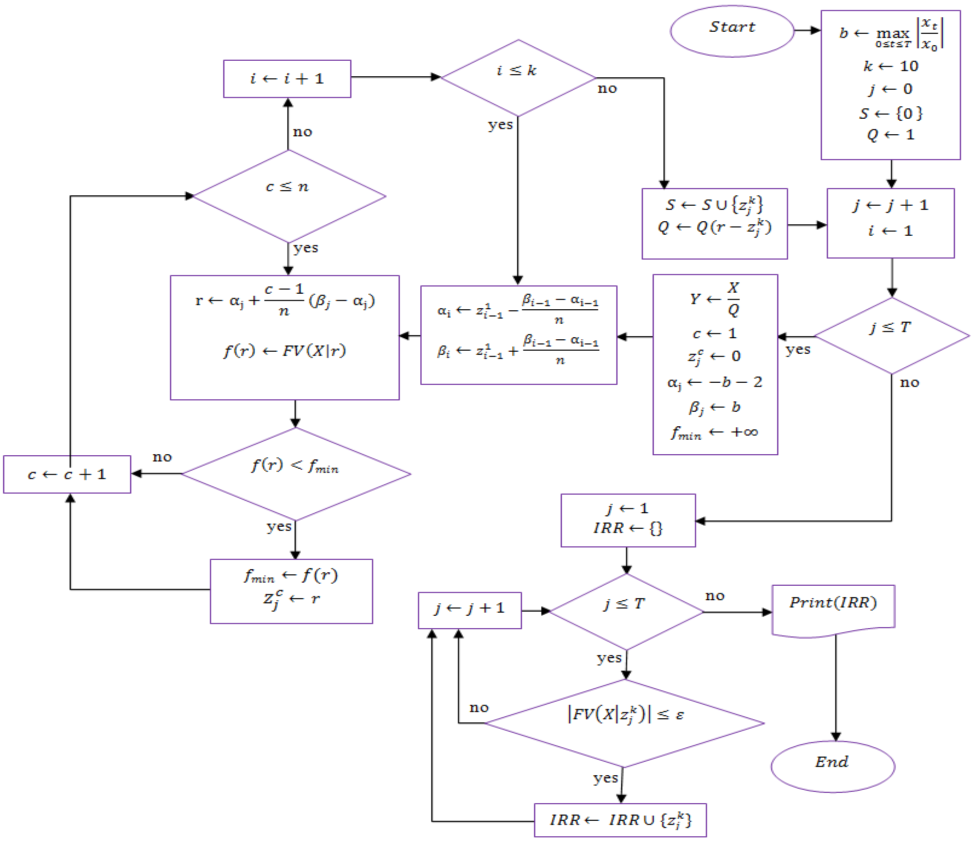

Fig. 1 Full flow chart of the steps of the proposed algorithm

Example (1): In this example, we intend to calculate the internal returns of cash flow X as follows using the proposed method for 10 repetitions:

X=(100000000, -900000000, 3570000000, -8190000000, 11972730000, -11563650000, 7377218000, -2997054000, 703404576, -72648576)

The cash flow in this example includes 9 real return rates ({-0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4}). Figure 2 shows how the algorithm converges from the first step to the tenth step for all the rates. Each color is related only to one rate.

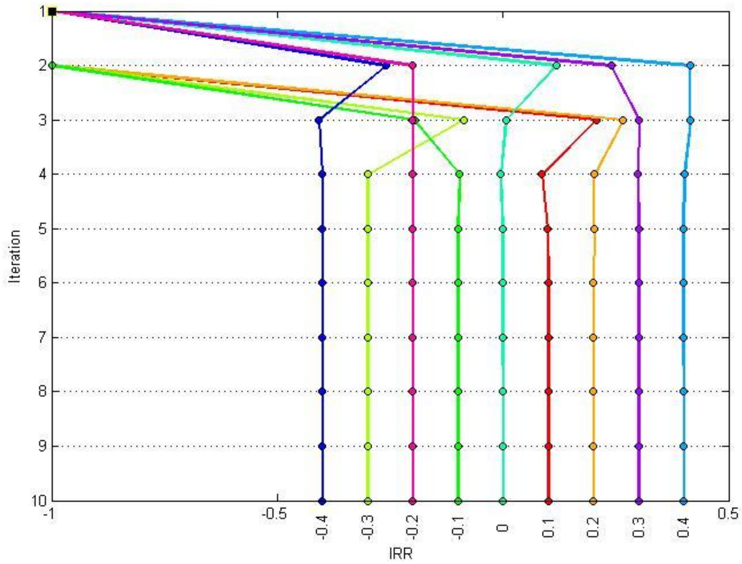

Fig. 2 The method of algorithm converges from the first step to the tenth step for all the rates in the form of different colors

As shown in Figure 2, all rates initially have a value of zero, and then in the next steps, the algorithm is closer to its actual value. Fixed points in the final steps indicate the convergence of the rates to their real value. The future value for all return rates is equal to zero, indicating the correctness of the results. These values are given in Table 1.

Table 1 Future values for all return rates

IRR	−0.4	−0.3	−0.2	−0.1	0	0.1	0.2	0.3	0.4
FV	0	0	0	0	0	0	0	0	0

The chart below shows the future value for the cash flow. The green circular points in Figure 3 are the internal rates of return. It is worth mentioning that perhaps the human can somewhat approximate the return rates visually through drawing charts; but in our proposed approach, there is no need for visual sense, and all steps of the algorithm are performed through numerical computations.

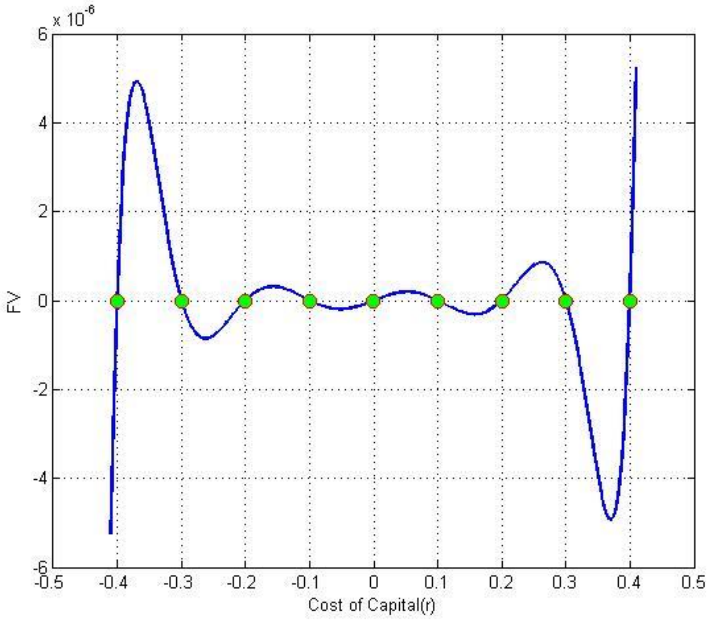

Fig. 3 Chart of future cash flow X

Example (2): In this example, our purpose is to calculate the internal returns of cash flow *X* as follows using the proposed method for 10 iterations:

X= (5000,-3000, 6000,-5000,-1000)

The cash flow examined in this example includes four return rates of {0.8631, -0.1642, -0.0494, -0.0494}, some of which are incorrect and should be discarded. For this purpose, the future value of the cash flow *X* is calculated for all the rates obtained. If the future value is far from zero, then we will not accept the corresponding return rate. This is shown in Table 2.

Table 2 The future value for all return rates

IRR	0.8631	−0.1642	−0.0494	−0.0494
FV	0	0	−737.7972	−737.7973
Decision	accept	accept	reject	reject

Figure 4 also shows the method of algorithm converges from the first step to the tenth step for all the rates.

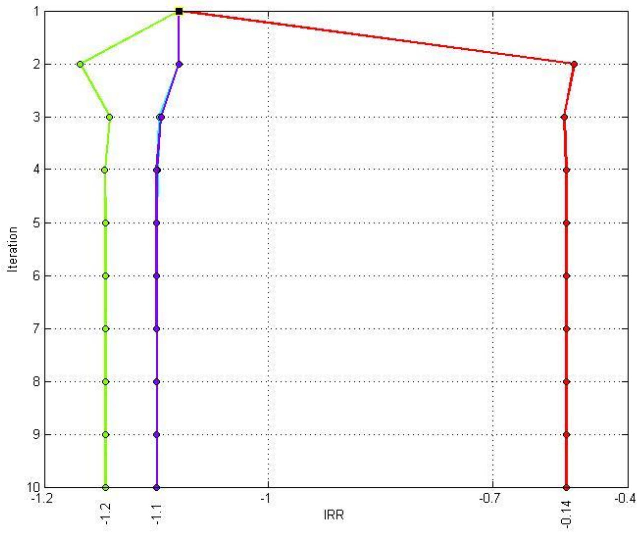

Fig. 4 Convergence of algorithm from the first step to the tenth step for all the rates in the form of different colors

Example (3): In this example, we intend to calculate the internal returns of cash flow *X* as follows using the proposed method for 10 repetitions:

X=(100, -220, 121)

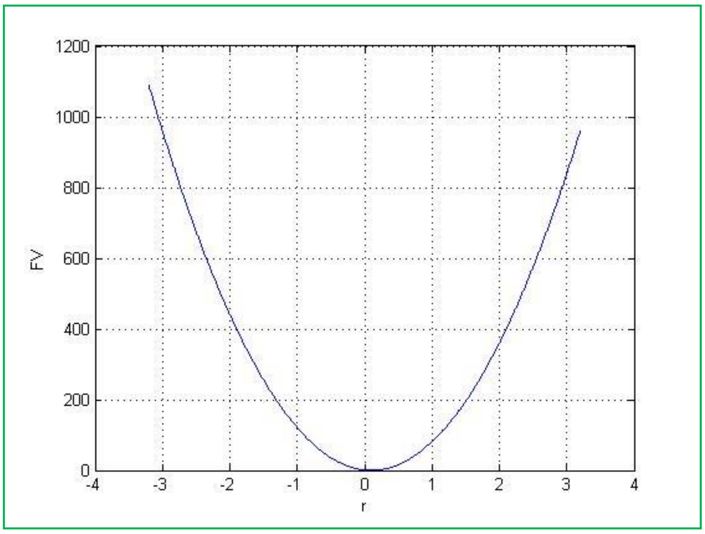

Fig. 5 Chart of future cash flow X

The cash flow in this example includes two return rates of {0.1, 0.1} that the repetitiveness of these rates is not seen in Figure 5. Also, it is not possible to determine through the eye the exact amount of return rate by the future value chart (i.e. Figure 5).

7 Discussion

There are various methods for calculating the root of functions, including Newton's method, Secant algorithm, and drawing methods. Major weaknesses are associated with these: 1. If the initial predictions are not precise enough, the algorithms may converge to a point that does not have a zero at all. 2. These methods cannot determine whether the zero is iterated. 3. If successful, these algorithms detect only one zero. 4. In the first two methods (Newton's method and Secant algorithm), differentiability is a necessary condition. Whereas, the method proposed in this study does not require differentiation, can determine iterated zeros, and the function value per discovered root is zero.

8 Conclusion

The Internal Rate of Return (IRR) method has a special place among economists and managers. According to its structure, this method has a close connection with the algebraic problems in classical mathematics, which makes it difficult to do related calculations. Accordingly, it was tried in the present study to help some researchers and activists in the area by introducing a relatively simple method. The method proposed in this paper has no need for differentiation, drawing the graph of the function, and the visual sense. The results indicate that the proposed method can calculate all real rates. It is worth noting that the proposed method is easy to understand and its execution by different software is by no means complicated.

Future studies can calculate the rate of return for gray and fuzzy cash flows.

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