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Efficiency analysis in multi-period system using DEA-R models

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Abstract In measuring the efficiency of a set of units in a time span that covers several periods, the models based on the standard DEA consider the system as a black box and ignore the status of each unit in each period, which causes misleading results. On the other hand, Wei et al. [14] showed that standard DEA models not only underestimate the efficiency score of inefficient DMU, but also identify efficient DMU as inefficient. In order to solve the above deficiencies, this paper develops DEA-R models by applying MOLP techniques in the presence of multi-period data in such a way that the proposed method can evaluate the overall efficiency according to the periodic efficiency of all units. The proposed method is a general method for p-periodic system. To clarify the details of the proposed method, a comparison between the existing models and the proposed multi-period DEA-R model has been made to measure the efficiency of 22 Taiwanese commercial banks in the period of 2009-2011.

Keyword: Ratio Data Envelopment Analysis (DEA-R), Multi-Periodic Production Process, Overall Efficiency, Pseudo-Inefficiency, Multi-Objective Linear Programming (MOLP).

1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric method that evaluates the relative efficiency of homogeneous units with multiple inputs and multiple outputs compared to each other. For the first time, Farrell in 1957 determined the efficiency in a non-parametric way. Charnes [1] extended Farrells work and the result of their work as the CCR model was published in 1978. Banker [2] actually developed Charnes et al.'s work by introducing the BCC model. Later, it was found that this technique is used in various fields, for example, in

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profit-driven companies such as banks [3], industry [4], hospitals [5], and retail stores [6], uncertainty environments [7] and other applications.

DEA is a standard technique for performance measurement. For cases in which the period of time being examined is composed of clearly defined time units such as years, the total inputs consumed and total outputs produced in all of the periods are aggregated for efficiency measurement. In 1999, Nemoto and Goto [8] presented a dynamic method to evaluate the efficiency of a multi-period system. Kao [9] used dynamic models to study systems with series structure. Ton and Tsutsui [10] were also other developers of this method in the study of network structures. Mariz [11] reviewed dynamic models and their application in various studies. The common point in dynamic DEA studies was that to calculate the total efficiency, the total inputs consumed and the total outputs produced in all periods are considered to measure the efficiency. The total efficiency calculated using the data used in the whole period only gives the total efficiency of the unit under evaluation (DMU_o) without considering the periodic efficiency of specific periods, which is one of the shortcomings of these methods. To involve period efficiency in the calculation of multi-period efficiency, the multi-period data envelopment analysis method (Multi Period DEA) was presented by Kao and Liu [12]. They proposed a relational network model that simultaneously calculates total efficiency and periodic efficiency. Their main focus in the model is on unit performance in each period to calculate efficiency. It is interesting to note that the overall efficiency is obtained by the weighted average of periodic efficiencies and the weights used are the most favorable weights in DMU under evaluation. Case studies on 22 commercial banks in Taiwan for 3 years from 2009 to 2011 indicate the power of differentiation of their proposed model compared to dynamic models. But according to the overall efficiency results obtained from Kao and Lius method in the evaluation of Taiwan banks, this method has identified all units as inefficient, which is a little thought-provoking. Recently, Wei et al. [13] tried to provide efficiency measurement in multi-period network DEA model with feedback, they used a binary heuristic algorithm to obtain optimal efficiency. But in the end, the relationship between overall efficiency and periodic efficiency still remains as a challenge. In 2011, Wei [14], showed that most DEA models, such as CCR, which are based on $\frac{\sum uy}{\sum vx}$ or $\frac{\sum vx}{\sum uy}$ cause two types of problems: Weak efficiency and pseudo inefficiency. Weak efficiency is the misclassification of inefficient DMUs as efficient DMUs. This deficiency is solved by the two-phase method [15] or the SBM model [16]. However, pseudo inefficiency, which identifies an efficient DMU as an inefficient DMU, is a neglected issue. In practice, pseudo inefficiency may lead to some misleading. An efficient hospital, after using CCR to evaluate its efficiency, may implement unnecessary policies or lose its strengths. Since pseudo inefficiency is a theoretical defect that leads to practical effects, Wei [14] investigated and identified pseudo inefficiency in a study in order to avoid unreasonable results. Reviewing other studies on the issue of weight constraints, they concluded that CCR not only underestimates the efficiency score of inefficient DMUs, but also identifies efficient DMUs as inefficient. Since this mistake, which they called as pseudo inefficiency, is not visible, they compared CCR-I with the assumption of weight restriction with DEA-R-I without the assumption of weight restriction and proved that the efficiency score of DEA-R-I is always greater than the CCR-I efficiency score. Then by comparing both methods to evaluate the performance of medical centers in Taiwan, they identified the units that had pseudo inefficiency and showed that the cause of pseudo inefficiency is the number of weights and also the assumption of weight restriction in CCR. On the other hand, in many organizations and financial institutions, it is in many cases more cost and time efficient to access ratio data.

Therefore, it is of great importance to evaluate the performance of decision-making units (DMUs) which only have access to ratios of inputs to outputs or vice versa (for instance, ratio of employees to students, ratio of assets to liabilities and ratio of doctors to patients). Therefore, it seems necessary to use DEA-R models instead of standard DEA models in practical approaches including multi-stage network approaches.

The idea of using data envelopment analysis model based on ratio analysis was proposed for the first time by Dispic et al. [17] and it was called DEA-R. In DEA, the coverage and multiple models are used in the nature of input and output with efficiency on a fixed and variable scale. According to the definition of efficiency, positive weights should be considered, this itself causes a weight limitation. On the other hand, by specifying the false efficiency scale in the data coverage analysis and presenting a suitable model, the real efficiency of the decision-making unit can be considered by considering the weighted sum of the ratio of each output to the input. Therefore, it seems necessary to change data coverage analysis models from the classical mode to data coverage analysis models based on fractional analysis. Dispic and colleagues [17] used the linear programming model by considering all the relationships formed between all outputs and all inputs for efficiency analysis and for the first time presented the DEA-R model to evaluate the efficiency of a unit. By introducing data coverage analysis models based on fractional analysis, they obtained the relationship between arithmetic, geometric and weighted mean in the efficiency value. He and colleagues [14, 18, 19] developed the approach of DEA-R models. Using DEA-R models, they evaluated 21 medical centers in Taiwan and investigated false inefficiencies. Li et al. [20] investigated DEA-R models without using explicit inputs in 15 Chinese research institutes. They presented a different approach focusing on defining the production possibility set and measuring technical efficiency. Based on these foundations, they developed input-oriented DEA-R models assuming constant returns to scale to evaluate efficiency and hyperefficiency.

DEA-R models were first formulated in Despic et al. [17] as a tool that combines DEA and ratio analysis, and since then, such models have been studied and applied by many other researchers. By employing DEA models on ratio-based data, the authors found the relationship between arithmetic mean, geometric mean, and weight in efficiency value. Wei et al. [14, 18, 19] extended the theory of DEA-R models. They focused on relations between traditional DEA models and ratio-based DEA-R models and applied the DEA-R models for an efficiency analysis of 21 medical centers in Taiwan. The authors analyzed Pseudo-inefficiency in these units. DEA-R models without explicit inputs were studied and verified in a case of 15 research institutes in China in Liu et al. [20]. They offered a different approach which focuses on the definition of the production possibility set and technical efficiency measurement. Based on this axiomatic foundation, they developed the input-oriented DEA-R models with the assumption of constant return to scale to evaluate efficiency and super efficiency. Cost and revenue efficiency in DEA and DEA - R models and the relationship between DEA models without explicit input and DEA-R are discussed in Mozaffari et al. [21, 22]. Mozaffari et al. [23] discussed the axioms for specifying the production possibility set in constant returns to scale technology for DEA-R, and, finally an original algorithm for identification of efficient surfaces in this class of models is proposed. Olesen et al. [24] demonstrated the problems with ratio data after classifying them, defined a production possibility set and introduced the corresponding models in constant/variable returns to scale technology and provided a positive answer to the existing debate with regard to the use of DEA models for ratio data. Olesen et al. [25] also discussed the method by which DEA models are solved with ratio data and introduced a new type of potential ratio (PR) inefficiency.

Recently, Kamyab et al. [26] developed CRA models based on DEA-R to evaluate commercial banks in a two-stage system. The results show that the proposed method obtains more accurate efficiency measures and therefore allows better discrimination between DMUs. Mozafari et al. [27] introduced a DEA-R based approach to consider managerial preferences. They presented a multi-objective linear programming (MOLP) model to evaluate the efficiency based on the definition of the production possibility set in the presence of ratio data and to obtain the corresponding pattern for each decision unit. All of these and other researches, in addition to their real-world applications, demonstrate the importance of this topic in the DEA literature.

The aim of this paper is to develop a multi-period production system, based on the DEA-R approach, to measure the overall efficiency of a set of DMUs in a period of time. To do this, we first propose a multi-objective model for a system with two time periods, then by linearizing the model, we generalize it to the general state of the p-period system. To emphasize the strengths of the proposed model, the proposed model is implemented on the data of 22 commercial banks in Taiwan and compared with the existing models.

The paper is organized as follows. The following section reviews the basic concepts of multi-period production system and a brief summary of DEA-R. In Section 3 the proposed approach for dealing with a multi - period system based on DEA-R models is introduced. Section 4 illustrates the applicability of the proposed method with a real numerical example. The conclusion will end the paper.

2 Preliminaries

2.1 Multi-period efficiency measure

Evaluating efficiency in multi-period models has attracted considerable attention among researchers**.** To describe the DEA efficiency measurement, assume there are *n DMUs* and the performance of each *DMU* is characterized by a production process of m inputs X_{ij} $(i=1,\dots,m)$ to yield s outputs Y_{rj} $(r=1,\dots,s)$. Consider a multi-period system composed of q periods, as shown in Fig. 1, where the superscript $p(p=1,\dots,q)$ in $X_{ij}^{(p)}$ and $Y_{rj}^{(p)}$ denotes the corresponding period. The total quantities of the i-th input and r –th output for $DMU_j(j=1,..,n)$ in all q periods are $X_{ij}=\sum_{p=1}^{q}X_{ij}^{(p)}$ and $Y_{rj}=\sum_{p=1}^{q}Y_{rj}^{(p)}$.

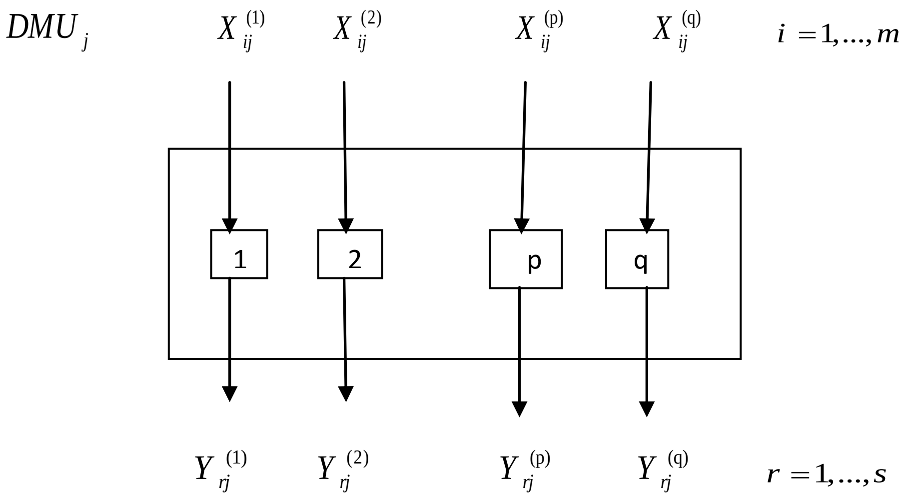

Fig. 1 The structure of multi period system.

Kao [9] and Oleson et al. [25] have conducted the standard CCR model (1) to evaluate the efficiency of a particular period $p(p=1,\dots,q)$ separately using the data for that period to Model (1). The CCR model measures the efficiency of DMU_k is as follows:

$$
\begin{aligned}
& E_k^{CCR} = Max \sum_{r=1}^{s} u_r Y_{rk} \\
& s.t. \\
& \sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \le 0 \quad , \; j=1,\dots,n \qquad (1) \\
& \sum_{i=1}^{m} v_i X_{io} = 1 \\
& u_r \ge 0, v_i \ge 0 \quad r=1,\dots,s \;\; , \; i=1,\dots,m.
\end{aligned}
$$

This model is a constant return to scale (CRS) program and u_r, v_i are the corresponding weights of the r –th output and the i-th input, respectively. Since, employing the total input $X_{ij} = \sum_{p=1}^{q} X_{ij}^{(p)}$ and the total output $Y_{rj} = \sum_{p=1}^{q} Y_{rj}^{(p)}$ in the time span to evaluate the overall efficiency of a system by the CCR model (1) ignoring the operations of individual periods, the Aggregate model was adopted to calculate the overall efficiency of a unit in a period of time. The model has the following format:

$$
\begin{aligned}
& E_k^{AGR} = Min \quad \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
& s.t. \\
& \qquad \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = \theta X_{ik}, \quad i = 1,\ldots,m \\
& \qquad \sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = Y_{rk}, \qquad r = 1,\ldots,s \\
& \qquad \lambda_j, s_i^-, s_r^+ \geq 0 \quad j = 1,\ldots,n, \;\; i = 1,\ldots,m, \;\; r = 1,\ldots,s
\end{aligned} \tag{2}
$$

The above Model (2) only calculates the overall efficiency of a DMU in a period of time. As for the treatment of individual periods into consideration in measuring the overall efficiency of q periods, Park and Park [28] extended model (2) through extensions of the concept of Farrells technical efficiency.

$$
\begin{aligned}
& E_k^{PP} = \min \quad \theta - \varepsilon\left(\sum_{p=1}^{q}\sum_{i=1}^{m} s_i^{-(p)} + \sum_{p=1}^{q}\sum_{r=1}^{s} s_r^{+(p)}\right) \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j^{(p)} X_{ij}^{(p)} + s_i^{-(p)} = \theta X_{ik}^{(p)}, \;\; p = 1,..q, \;\; i = 1,\ldots,m \\
& \sum_{j=1}^{n} \lambda_j^{(p)} Y_{rj}^{(p)} - s_r^{+(p)} = Y_{rk}^{(p)}, \quad p = 1,\ldots,q, \;\; r = 1,\ldots,s \\
& \lambda_j^{(p)}, s_i^{-(p)}, s_r^{+(p)} \geq 0, \;\; p = 1,\ldots,q, \;\; r = 1,\ldots,s \;, \;\; i = 1,\ldots,m, \; j = 1,\ldots,n
\end{aligned} \tag{3}
$$

It should be pointed out that Model(2) is a special case of model (3) with the intensity variable $\lambda_j^p (p = 1,..,q, j = 1,..n)$ for each period as independent process is modeled through the use of slack variables in the constraints. Notably, the Model (3) is the adaptation of the network DEA model of Fare and Grosskopf [29] for the system shown in Fig.1. Since, these periods are connected with a unique distance measure of θ, the Model(3) is called the connected network model. The overall efficiency measure is the distance measure of the best-performing period adjusted by ε effect of slack variables and regarding to this effect a DMU is overall efficient only if it is efficient in all the periods. As for treatment of Fig.1, Kao and Liu [12] have developed the relational network model, based on the condition that if each period is viewed as a process of a network system, then it resembles the structure of a parallel system with q processes. The model has the following format:

$$
\begin{aligned}
&E_k^{KL} = \max \sum_{r=1}^{s} u_r Y_{rk} \\
&s.t \\
&\sum_{i=1}^{m} v_i X_{ik} = 1 \\
&\sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \le 0, j = 1,...,n \\
&\sum_{r=1}^{s} u_r Y_{rj}^{(p)} - \sum_{i=1}^{m} v_i X_{ij}^{(p)} \le 0, j = 1,...,n, p = 1,...,q \\
&u_r, v_i \ge \varepsilon, r = 1,...,s, i = 1,...,m
\end{aligned} \tag{4}
$$

The main characteristics of model (4) can be stated as follows. First, in this model (4), the weights related to similar factors are identical with respect to the corresponding period. Second, not only inputs and outputs but also their corresponding periods are considered in calculating the overall efficiency of the multi-period system. Applying the optimal solutions u_r^*, v_i^* , the overall efficiency $E_{overall}$ and each period efficiency $E_k^p (p=1,...,q)$ are calculated as follows:

$$
E_{overall} = \frac{\sum_{r=1}^{s} u_r^* Y_{rk}}{\sum_{i=1}^{m} v_i^* X_{ik}} = \sum_{r=1}^{s} u_r^* Y_{rk} \tag{5}
$$

$$
E_i^{(p)} = \frac{\sum_{r=1}^{s} u_r^* Y^{(p)}_{rk}}{\sum_{i=1}^{m} v_i^* X^{(p)}_{ik}} \ , \quad p = 1,...,q \tag{6}
$$

On the other hand, the results of applying model (4) on 22 Taiwanese commercial banks in the period of 2009-2011 in the article by Kao and Liu [12] indicate that due to the existence of pseudo inefficiency caused by the application of the CCR, this model has evaluated all units as inefficient. Wei et al. [14] showed that CCR not only underestimates the efficiency score of inefficient DMU, but also identifies efficient DMU as inefficient. In this article, to solve this problem, we expand the DEA-R models in the multi-period space and introduce a new model that calculates the overall efficiency of units by considering the efficiencies of all units in all periods and produce the reasonable and acceptable efficiency measure.

2.2 DEA-R models

Again, suppose that there are n DMUs and for $DMU_j (j=1,...,n)$ the observed data of inputs and outputs are $X_j = (x_{1j},....,x_{mj}) > 0$ and $Y_j = (y_{1j},..., y_{sj}) > 0$. Also assuming the ratios $\frac{x_{ij}}{x_{io}}$

and $\frac{y_{rj}}{y_{ro}}$ are defined. Despic et al.[17] have introduced their DEA-R efficiency model for evaluation of DMU_o under the assumption of constant returns to scale technology as follows:

$$\begin{aligned}
\hat{e}_0 = Max_{w_{ir}}\, Min_j \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}\, \frac{(X_{ir}/Y_{rj})}{(X_{io}/Y_{ro})} \\
s.t. \quad \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir} = 1, \\
w_{ir} \geq 0 \quad i=1,\dots,m \;\; r=1,\dots,s
\end{aligned} \tag{7}$$

The model assumes that x_{io} and y_{ro} are the input and output vectors of DMU$_o$, and w_{ir} represents the relative weight of i-th input and r-th output of input and output vector variables.

Definition 1. The under evaluated unit (DMU_o) is efficient if and only if the optimal objective function value of model (7) i.e., $\hat{e}_o^* = 1$, otherwise it is inefficient.
Input-oriented and output-oriented models of DEA-R model (7) are defined as follows in the case of constant returns to scale [30].

$$\begin{aligned}
\max \quad & \theta \\
s.t. \quad & \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir} \frac{(X_{ij}/Y_{rj})}{(X_{io}/Y_{ro})} \geq \theta, \qquad j=1,\dots,n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir} = 1, \\
& w_{ir} \geq 0 \quad i=1,\dots,m \;\; r=1,\dots,s
\end{aligned} \tag{8}$$

$$\begin{aligned}
\min \quad & \varphi \\
s.t. \quad & \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir} \frac{(Y_{rj}/X_{ij})}{(Y_{ro}/X_{io})} \leq \varphi, \qquad j=1,\dots,n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir} = 1, \\
& w_{ir} \geq 0 \quad i=1,\dots,m \;\; r=1,\dots,s
\end{aligned} \tag{9}$$

Model (8) is the input-oriented DEA-R model and model (9) is the output-oriented DEA-R model. Both models are standard linear programming problems.

3 Proposed method

3.1 Two period DEA-R model based on MOLP

Suppose X_{ij} and Y_{rj} are respectively the i-th input (i=1,...,m) and the r-th output (r=1,...,s), DMU_j (j=1,...n) and the system consists of 2 periods, which the first period consumes the input $x_{ij}^{(1)}$ to produce the output $y_{rj}^{(1)}$ and the second period consumes the input $x_{ij}^{(2)}$ to produce the output $y_{rj}^{(2)}$. Using the DEA-R model (8), the efficiency of the units in the input-oriented model is calculated in each period. In other words, models (10) and (11) calculate the efficiency of the unit under evaluation in the first and second periods, respectively.

$$
\begin{aligned}
\max\ & \theta^{(1)} \\
s.t.\ & \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(1)} \frac{(X_{ij}^{(1)}/Y_{rj}^{(1)})}{(X_{io}^{(1)}/Y_{ro}^{(1)})} \geq \theta^{(1)}, \qquad j=1,\dots,n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(1)} = 1, \\
& w_{ir}^{(1)} \geq 0 \quad i=1,\dots,m \;\; r=1,\dots,s
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\max\ & \theta^{(2)} \\
s.t.\ & \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(2)} \frac{(X_{ij}^{(2)}/Y_{rj}^{(2)})}{(X_{io}^{(2)}/Y_{ro}^{(2)})} \geq \theta^{(2)}, \qquad j=1,\dots,n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(2)} = 1, \\
& w_{ir}^{(2)} \geq 0 \quad i=1,\dots,m \;\; r=1,\dots,s
\end{aligned}
\tag{11}
$$

In order to obtain the overall efficiency of the unit under evaluation, the following multi-objective model can be used.

$$
\begin{aligned}
\max\ & \left\{\theta^{(1)}, \theta^{(2)}\right\} \\
s.t.\ & \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(1)} \frac{(X_{ij}^{(1)}/Y_{rj}^{(1)})}{(X_{io}^{(1)}/Y_{ro}^{(1)})} \geq \theta^{(1)}, \qquad j=1,\dots,n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(2)} \frac{(X_{ij}^{(2)}/Y_{rj}^{(2)})}{(X_{io}^{(2)}/Y_{ro}^{(2)})} \geq \theta^{(2)}, \qquad j=1,\dots,n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(1)} = 1, \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(2)} = 1, \\
& w_{ir}^{(1)},\ w_{ir}^{(2)} \geq 0 \quad i=1,\dots,m \;\; r=1,\dots,s
\end{aligned}
\tag{12}
$$

To solve the multi-objective model (12), the weighted sum method of the objective function can be used. In this regard, by considering positive parameters ρ_1 and ρ_2 (with the condition $\rho_1+\rho_2=1$) for the first and second objective function, respectively, model (12) becomes a one-objective linear programming problem as model (13) that depends on the parameters ρ_1 and ρ_2.

$$
\begin{aligned}
\max \quad & \rho_1\theta^{(1)} + \rho_2\theta^{(2)} \\
s.t. \quad & \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(1)} \frac{\left(x_{ij}^{(1)}/y_{rj}^{(1)}\right)}{\left(x_{io}^{(1)}/y_{ro}^{(1)}\right)} \geq \theta^{(1)}, \qquad j = 1, \dots, n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(2)} \frac{\left(x_{ij}^{(2)}/y_{rj}^{(2)}\right)}{\left(x_{io}^{(2)}/y_{ro}^{(2)}\right)} \geq \theta^{(2)}, \qquad j = 1, \dots, n \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(1)} = 1, \\
& \sum_{i=1}^{m}\sum_{r=1}^{s} w_{ir}^{(2)} = 1, \\
& w_{ir}^{(1)}, \quad w_{ir}^{(2)} \geq 0 \quad i = 1, \dots, m, r = 1, \dots, s
\end{aligned} \tag{13}
$$

The input-oriented DEA-R envelopment model for evaluation of the DMU_o can be formulated as follows:

$$
\begin{aligned}
\alpha_o = \min \quad & \alpha^{(1)} + \alpha^{(2)} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j^{(1)} \left(\frac{X_{ij}^{(1)}}{Y_{rj}^{(1)}}\right) \leq \alpha^{(1)} \left(\frac{X_{io}^{(1)}}{Y_{ro}^{(1)}}\right), \qquad i = 1, \dots, m, r = 1, \dots, s \\
& \sum_{j=1}^{n} \lambda_j^{(2)} \left(\frac{X_{ij}^{(2)}}{Y_{rj}^{(2)}}\right) \leq \alpha^{(2)} \left(\frac{X_{io}^{(2)}}{Y_{ro}^{(2)}}\right), \qquad i = 1, \dots, m, r = 1, \dots, s \\
& \sum_{j=1}^{n} \lambda_j^{(1)} = \rho_1, \qquad \sum_{j=1}^{n} \lambda_j^{(2)} = \rho_2 \\
& \rho_1 + \rho_2 = 1 \\
& \lambda_j^{(1)}, \lambda_j^{(2)} \geq 0 \qquad j = 1, \dots, n
\end{aligned} \tag{14}
$$

Model (14) is an input-oriented linear programming problem based on ratio analysis. As it comes from the constraints of the model, all the data have been used as ratio data in the model. The variables $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ correspond to the first and the second periods, respectively. If $\sum_{j=1}^{n}\lambda_j^{(2)} = 0$ (i.e. $\rho_1 = 1$ and $\rho_2 = 0$) then only first period is considered and the overall efficiency of the system is the same as the efficiency of period 1. Similarly, if $\sum_{j=1}^{n}\lambda_j^{(1)} = 0$ (i.e. $\rho_1 = 0$ and $\rho_2 = 1$) we have the same for second period. By adopting $\sum_{j=1}^{n}\lambda_j^{(1)} = \sum_{j=1}^{n}\lambda_j^{(2)} = 1/2$ (i.e. $\rho_1 = \rho_2 = 1/2$) the efficiency of the whole system, which is the average efficiency of the unit under evaluation in two periods is obtained. In general, if $\sum_{j=1}^{n}\lambda_j^{(1)} = \rho_1$ and $\sum_{j=1}^{n}\lambda_j^{(2)} = \rho_2$ are considered such that $\rho_1 + \rho_2 = 1$ and $\rho_1, \rho_2 > 0$ then the optimal solution of model (14) defines the overall efficiency of the unit under evaluation in the two-period system. By changing the value of parameters, the overall efficiency value changes. Therefore, determining the value of the parameters plays an important role in calculating the overall efficiency of the decision-making unit.

3.2 The extension of the proposed DEA-R model in the P-period system

Suppose the system consists of q periods, which the p-th period consumes input $X_{ij}^{(p)}$ to produce the output $Y_{rj}^{(p)}$. Consider Fig 1 again. In the following, a general model for calculating the overall efficiency of a multi-period production process based on DEA-R models is introduced. To do this, consider the variables $\lambda_j^{(1)}, \lambda_j^{(2)}, ..., \lambda_j^{(q)}$ corresponding to periods 1, 2, ..., q respectively. The proposed input-oriented DEA-R model for the q-period system under the assumption of constant returns to scale is as follows:

$$
\begin{aligned}
&\alpha_o = \min \sum_{p=1}^{q} \alpha^{(p)} \\
&\quad s.t. \ \sum_{j=1}^{n} \lambda_j^{(p)} \left(\frac{X_{ij}^{(p)}}{Y_{rj}^{(p)}} \right) \le \alpha^{(p)} \left(\frac{X_{io}^{(p)}}{Y_{ro}^{(p)}} \right), \qquad i = 1,...,m, r = 1,...,s; p = 1,...,q \\
&\qquad \sum_{j=1}^{n} \lambda_j^{(p)} = \rho_p, \quad p = 1,...,q \\
&\qquad \sum_{p=1}^{q} \rho_p = 1 \\
&\qquad \lambda_j^{(p)} \ge 0 \qquad\qquad j = 1,...,n; \ p = 1,...,q
\end{aligned}
\tag{15}
$$

It is notable that the optimal value of the objective function in model (15) calculates the overall efficiency of the unit under evaluation according to the efficiency of the unit in all periods.

The proposed model has the following features:

1. The weights related to the same factors are different compared to the relevant period.
2. The model is always feasible and the optimal solution of the objective function is always between zero and one. (To see the proof, refer to the appendix).
3. The model is unit invariant.
4. If access to ratio data is very affordable in terms of cost and time, only DEA-R models can calculate the efficiency measure and DEA models cannot determine the efficiency measure. The proposed model is applicable to both ratio data and normal data.
5. Model (15) calculates the overall efficiency of the unit under evaluation according to the efficiencies of all periods.

Definition 2. The unit under evaluation is overall efficient if and only if the optimal value of the objective function in model (15) is equal to 1 for positive ρ, that is, $\alpha_o^* = 1$ otherwise the unit under evaluation is inefficient.

3.3 Calculating the positive parameters to determine the overall efficiency in the p-period system

From the model (15), it appears that positive parameter values play an important role in calculating the overall efficiency of the units. The suggested formula for calculating the values ρ_p in DMUo $(o = 1,...,n)$ is as follows:

$$\rho_p = \frac{\sum_{o=1}^{n} (\alpha_o^{*(p)} - \overline{\alpha}_0^*)}{\sum_{p=1}^{q} \sum_{o=1}^{n} (\alpha_o^{*(p)} - \overline{\alpha}_0^*)} \tag{16}$$

In this equation

1. $\alpha_o^{*(p)}$ is the efficiency of the unit under evaluation in the p-th period, in other words, the optimal solution of the model (15) for $\sum_{j=1}^{n} \lambda_j^{(k)} = 0 \quad (k \neq p)$ and $\sum_{j=1}^{n} \lambda_j^{(p)} = 1$.
2. $\overline{\alpha}_o^*$ is the measure of the efficiency of the unit under evaluation by considering the total input of all periods as total input (i.e. $X_{ij} = \sum_{p=1}^{q} x_{ij}^{(p)}$) and the total output of all periods as total output (i.e. $Y_{rj} = \sum_{p=1}^{q} Y_{rj}^{(p)}$). In other words, by considering the entire system as a black box. the DEA-R model (15) is modified as follows:

$$\begin{aligned} \overline{\alpha_o} &= \min \alpha \\ s.t. &\sum_{j=1}^{n} \lambda_j \left(\frac{X_{ij}}{Y_{rj}}\right) \leq \alpha \left(\frac{X_{io}}{Y_{ro}}\right), \quad i = 1,...,m; \quad r = 1,...,s, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \geq 0 \qquad j = 1,...,n \end{aligned} \tag{17}$$

In short, it can be said that the comprehensive DEA-R model proposed in this article calculates the overall efficiency of the units by using the periodical efficiency of the units and allows the decision maker to have a logical prioritization with the identification of efficient and inefficient units and this is the biggest distinction of the proposed model in comparison with the existing approaches. It is worth mentioning that what has been said for a p-periodic system in the input-oriented mode, the process can be generalized for the output-oriented mode.

4 Numerical example

To show the applicability and the merits of the proposed method and meanwhile to compare it with the models (2), (3) and (4), a dataset of a real case consisting of 22 commercial banks taken from Kao and Liu [12] was used in this example. The data sets consist of three input factors (Labor, Physical capital and Purchased funds) and three output factors (Demand Deposits, S-term Loans and ML-term Loans). The data set are recorded over three time periods (2009,2010,2011). Employing Models (2), (3) and (4) the overall efficiency are calculated. The results are shown in Table 1.

Table 1 The results of the overall efficiency by applying models (1), (2) and (3)

Banks	Aggregate model(2)	Connected network model(3)	Relation network model(4)
1.Chang Hwa	0.9362	0.9472	0.8981
2.Kings Town	0.7809	0.8060	0.7457
3.Taichung	1.0000	1.0000	0.9721
4.Taiwan Business	1.0000	0.9988	0.9681
5.Kaohsiung	1.0000	1.0000	0.9731
6.Cosmos	0.7868	0.8113	0.7361
7.Union	0.5304	0.5635	0.5067
8.Far Eastern	0.8887	0.9963	0.7591
9.Ta Chong	0.7997	0.8653	0.7202
10.En Tie	0.9595	0.9997	0.9018
11.Hua Nan	1.0000	1.0000	0.9754
12.Fubon	1.0000	0.9979	0.9680
13.Cathay	0.8538	0.8629	0.8173
14.East Sun	1.0000	1.0000	0.9878
15.Yuanta	1.0000	1.0000	0.9475
16.Mega	1.0000	1.0000	0.9683
17.Taishin	0.6533	0.7865	0.5280
18.Shin Kong	0.8482	0.8615	0.8123
19.Sino Pac	0.9018	0.9433	0.8430
20.China Trust	0.8540	0.8881	0.6259
21.First	0.9592	0.9746	0.9279
22.Taiwan Cooperative	1.0000	1.0000	0.9818

The second column of Table 1 reports the efficiency calculations resulting from Aggregate model (2). The third column shows the overall efficiency calculated by the connected network model (3), and finally, the fourth column shows the overall efficiency using the relational network model (4). According to the information obtained from Table 1, units 3, 4, 5, 11, 12, 14, 15, 16 and 22 under the aggregation model and units 3, 5, 11, 14, 15, 16 and 22 under the connected network model are efficient. The fourth column clearly shows that the overall efficiencies obtained from the relational model of Kao and Liu [12] are less than 1 and also are less than or equal to the efficiencies obtained from the aggregation and connected network models. The results show that model (4) has evaluated all units as inefficient.

Table 2 shows the efficiency of each unit in all 3 periods using the proposed comprehensive model. The efficiency of the k-th period is obtained by considering $\rho_k = 1, \rho_j = 0, j \neq k$ in model (15). The second to fourth columns show the efficiency of the first to third periods, respectively.

Table 2 The optimal value of the objective function resulting from model (15), using optimal weights derived from equation (16)

	$\rho_1 = 1$ $\rho_2 = \rho_3 = 0$	$\rho_2 = 1$ $\rho_1 = \rho_3 = 0$	$\rho_3 = 1$ $\rho_1 = \rho_2 = 0$		$\rho_1 = 0.2267$ $\rho_2 = 0.3466$ $\rho_3 = 0.4267$
DMUs	$\boldsymbol{\alpha}^{*(\mathbf{1})}_{\boldsymbol{o}}$	$\boldsymbol{\alpha}^{*(\mathbf{2})}_{\boldsymbol{o}}$	$\boldsymbol{\alpha}^{*(\mathbf{3})}_{\boldsymbol{o}}$	$\bar{\boldsymbol{\alpha}}^{*}{}_{\boldsymbol{o}}$	$\alpha^{*}{}_{o}$
DMU01	0.9554	0.9339	0.9355	0.9399	0.9395
DMU02	0.7859	0.8108	0.7348	0.7893	0.7725
DMU03	1.0000	1.0000	1.0000	1.0000	1.0000
DMU04	0.9805	1.0000	1.0000	1.0000	0.9956
DMU05	۰.۹۹۹۸	1.0000	1.0000	1.0000	۰.۹۹۹۹
DMU06	0.7558	0.7989	0.8153	0.8104	0.7961
DMU07	0.5491	0.5695	0.4859	0.5349	0.5292
DMU08	0.8138	0.8413	1.0000	0.9017	0.9028
DMU09	0.8678	0.8069	0.7306	0.8003	0.7881
DMU10	1.0000	0.9873	1.0000	0.9592	0.9956
DMU11	1.0000	1.0000	1.0000	1.0000	1.0000
DMU12	0.9979	1.0000	1.0000	1.0000	0.9995
DMU13	0.8532	0.8805	0.8378	0.8653	0.8561
DMU14	1.0000	1.0000	1.0000	1.0000	1.0000
DMU15	1.0000	1.0000	1.0000	1.0000	1.0000
DMU16	1.0000	1.0000	1.0000	1.0000	1.0000
DMU17	0.7982	0.5670	0.5571	0.6686	0.6152
DMU18	0.8260	0.8654	0.8282	0.8481	0.8406
DMU19	0.8489	0.9298	0.9194	0.8927	0.9070
DMU20	0.9097	0.6873	0.7024	0.8860	0.7442
DMU21	0.9915	0.9883	0.9203	0.9683	0.9600
DMU22	1.0000	1.0000	1.0000	1.0000	1.0000

According to the information in Table 2, units 3, 10, 11, 14, 15, 16 and 22 in the first period, units 3, 4, 5, 11, 12, 14, 15, 16 and 22 in the second period and units 3, 4 , 5, 8, 10, 11, 12, 14, 15, 16 and 22 in the third period are efficient. The results recorded in the fifth column, which considers the whole system as a black box, indicate that units 3, 4, 5, 8, 10, 11, 12, 14, 15, 16 and 22 are all efficient. Using equation (16) to calculate ρs, we have: $\rho_1 = 0.2267,\ \rho_2 0.3466,\ \rho_3 = 0.4267$. The last column of Table 2 shows the overall efficiency of the units using the obtained ρs. The results show that only units 3, 11, 14, 15, 16 and 22 are overall efficient. According to the periodic efficiency of the units and the condition $\sum_{p=1}^{q} \rho_p = 1$ only units are overall efficient that the periodic efficiencies for these units are 1 in all periods. The obtained values for ρs are only effective for calculating and analyzing the efficiency of inefficient units. In practice, when the number of periods increases, the number of units that are efficient in all periods will be rarely. In such a situation, the proposed approach to analyze and evaluate the overall efficiency of the units is of great importance. According to the information in tables 1 and 2, the proposed model has a better ability to distinguish efficient and inefficient units, because under the aggregation models units 4 and 5 and under the connected network model unit 5, are efficient, but the proposed model introduced them as inefficient. The reason is that these units under the proposed DEA-R model are not efficient in all periods to be introduced as overall efficient. On the other hand, by comparing the results of the proposed approach with the relational network model, we can see that the efficiency size has increased and led to the identification of efficient units, which is the effect of using DEA-R models. In fact, the results show that units 3, 11, 14, 15, 16 and 22 in the relational network model had pseudo inefficiency because they were introduced as inefficient. In the end, it can be said that the calculation of periodic efficiency as well as the overall efficiency of units using a unit model and increasing the capability of the model in distinguishing efficient and inefficient units by using DEA-R models are the main advantages of the proposed model compared to the existing models.

5 Conclusion

The use of ratio-based data is an interesting and challenging issue in the Data Envelopment Analysis (DEA) literature, particularly in identifying the pseudo inefficiency in order to avoid common illogical results in classical models. On the other hand, many studies investigated how to measure the efficiency of a set of units over a period of time. When the time frame of efficiency measurement covers several periods, obtaining the overall efficiency of the units is a challenge that has been investigated by various authors. Some used the total data of all the periods to obtain the overall efficiency, and some others calculated the period efficiencies and considered their average as the overall efficiency. In response to the weakness of the existing multi-period models, a method based on DEA-R was proposed to measure the overall efficiency of the unit under evaluation by considering the efficiency of all units in all periods. In this study, MOLP techniques were used for a two-phase system, which after linearization was generalized to the general p-periodic system. Finally, an application on 22 Taiwanese commercial banks shows the practicality of the proposed model. In particular, the proposed approach, in addition to distinguishing between efficient and inefficient units, provides more logical results compared to existing approaches.

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Appendix

Theorem. Model (15) is always feasible and the optimal solution of the objective function is always between zero and one.

Proof. Suppose α_o^* is the optimal value of objective function of model (15);

a. If $\sum_{j=1}^{n}\lambda_j^{(p)}=0 \quad (2\le p\le q)$ then $\sum_{j=1}^{n}\lambda_j^{(1)}=1$. So, $\lambda_j^{(1)}=0 \quad (j\ne o)$, $\lambda_o^{(1)}=1, \alpha^{(1)}=1$ and $\alpha^{(p)}=0 \quad (2\le p\le q)$ is a feasible solution for model (15);

b. If $\sum_{j=1}^{n}\lambda_j^{(p)}=0 \ (1\le p\le q, p\ne 2)$ then $\sum_{j=1}^{n}\lambda_j^{(2)}=1$. So, $\lambda_j^{(2)}=0 \quad (j\ne o)$, $\lambda_o^{(2)}=1$, $\alpha^{(2)}=1$and $\alpha^{(p)}=0 \quad (1\le p\le q, p\ne 2)$ is a feasible solution for model (15);

c. As the same way, If $\sum_{j=1}^{n}\lambda_j^{(p)}=0 \ (1\le p\le q, p\ne k)$ then $\sum_{j=1}^{n}\lambda_j^{(k)}=1$. So, $\alpha^{(k)}=1, \lambda_o^{(k)}=1, \lambda_j^{(k)}=0 (j\ne o)$, and $\alpha^{(p)}=0 \ (1\le p\le q, p\ne k)$ is a feasible solution for model (15);

d. In general, if for any $1\le p\le q$, $\rho_p=\frac{1}{q}$, then from $\sum_{j=1}^{n}\lambda_j^{(p)}=\frac{1}{q}$ for $1\le p\le q$ we can conclude that $\lambda_j^{(p)}=0, j\ne o$, $\lambda_o^{(p)}=\frac{1}{q}$ and $\alpha^{(p)}=\frac{1}{q}$ is a feasible solution for model (15).

From this feasible solution, it can be concluded that the optimal value does not exceed one and is always greater than zero.