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Zero Weights in Weak Efficient and Inefficient Points with AHP

F. Hosseinzadeh Lotfi*, E. Ahanchi, M. Ahadzadeh Namin

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Abstract In this paper an approach in data envelopment analysis dealing with evaluation of non – zero slacks was propounded. This approach intends that weight of inefficient and weak efficient points that have been evaluated as zero weight, be considered as positive weight, and also in this approach, the pareto efficiency evaluates the picture of this point. In this approach positive weights in the inefficient and weak efficient points are appointed based on the decision maker's opinions and wants, and weights with help of analytical hierarchy process (AHP) are calculated and added to the model under studying, and gain a model with actions of weighted border.

Keywords Data Envelopment Analysis, Zero Weight, Analytical Hierarchy Process , Weak Efficient, Inefficient.

1 Introduction

The CCR model, introduced by Charnes et al. [1] is appointed as the beginning point of a new discussion in the realm of data envelopment analysis. Many opinions and views about control weights have been presented which are backgrounds for discussion about the aforesaid paper.

In most cases in practice, the DEA models assess the efficiency of the inefficient units by using reference points on the frontier of the production possibility set (PPS) that are not Pareto-efficient.

This happens as a result of the fact that these models usually yield zero weights for the optimal multipliers, or equivalently (by duality), strictly positive values for the optimal slacks, which means that the efficiency scores obtained for these units do not account for all sources of inefficiency. Bessent et al. [2] deal with the so-called ''not naturally enveloped inefficient units", which are defined as those that have a mix of inputs and/or outputs, different from that of any other point on the efficient frontier. The authors report the results corresponding to several studies that reveal the high frequency of the not naturally enveloped inefficiency units in practice.

These units are actually those in $F \cup NF$ according to the classification of the decision making units (DMUs) in Charnes et al. [3] (the DMUs in F are on the weak efficient frontier whereas those in NF are projected onto points in F). Much attention in the literature has been paid to this type of DMUs where we can find a wide variety of approaches intended to

* **Corresponding Author.** (✉)
E-mail: farhad@hosseinzadeh.ir (F. HosseinzadehLotfi)

F. HosseinzadehLotfi, E. Ahanchi
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran, P. O. Box: 14515-775.

M. AhadzadehNamin
Department of Mathematics, Shahr-e-qods Branch, Islamic Aazad University, Shahr-e-qods, Iran.

provide efficiency scores for them trying to avoid the problems with the non-zero slacks. An approach that is propounded uses the model of analytical hierarchy process and Assurance Region.

2 Background

2.1 Evaluation of weights in the analytical hierarchy process

In the analytical hierarchy process, first, elements are compared in the form of pair and paired comparison matrix, then formed by use of this matrix to calculate the relative weights of elements to tally a paired comparison matrix shown in the following form in which a_{ij} is the preference of element i to element j. Now with the determination of a_{ij}, we want to gain weights w_i of elements

$$A = [\, a_{ij} \,], \qquad i, j = 1, 2, ..., n.$$

Each paired comparison matrix may be consistent ($a_{ij} = \frac{w_i}{w_j}$) or inconsistent ($a_{ij} \neq \frac{w_i}{w_j}$) in the state that the matrix is consistent, and calculating weight w_i is simple and gained from normalization of the elements of each column.

But in the state that matrixes are inconsistent, four main approaches will be presented for calculation:

1 Least squares method.
2 Logarithmic least squares method.
3 Eigenvector methods.
4 Approximation methods.

Now we explain one of the above methods that we have used for achieving weights in the presented method.

In this method w_i is a determinant in a way that the following relationship is true.

$$\begin{aligned} a_{11}w_1 + a_{12}w_2 + \ldots + a_{1n}w_n &= \lambda.w_1 \\ a_{21}w_1 + a_{22}w_2 + \ldots + a_{2n}w_n &= \lambda.w_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{n1}w_1 + a_{n2}w_2 + \ldots + a_{nn}w_n &= \lambda.w_n \end{aligned}$$

That here a_{ij} is a preference of i-th element to j-th, and w_i is a weight of i-th element ,and λ is a constant number, this method is one kind of mean that Harker [4] calls it as possible mean in a different way. Because in this way the weight of i-th element (w_i) according to the above definition is equal to:

$$w_i = \frac{1}{\lambda}\sum_{j=1}^{n} a_{ij} w_j \qquad i = 1,2,\ldots,n$$

we can write the above simultaneous equations as follows:

$$A \times W = \lambda.W$$

In which A is a paired comparison matrix {mean $A = \left[a_{ij}\right]$}, and W is a weight vector, and λ is a scalar (number). According to the definition, this relationship is among one matrix

(A), vector (W) and (λ) number, it has been said that W is a special vector and λ is a special amount.

Example: if the paired comparison matrix be as follows, we can calculate the weight of criterion by using the model of special vector.

$$A = \begin{bmatrix} 1 & 1/3 & 1/2 \\ 3 & 1 & 3 \\ 2 & 1/3 & 1 \end{bmatrix}$$

Solution: $A = (A - \lambda I)$ we form the zero matrix.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1/3 & 1/2 \\ 3 & 1-\lambda & 3 \\ 2 & 1/3 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) + 5/2 = 0$$

is calculated $\lambda_{\max} = 3.0536$ After the solution of the above third_degree equation. w_i is calculated as: $(A - \lambda_{\max} I)W = 0$ Now we form the equation

$$\begin{bmatrix} -2.0536 & 1/3 & 1/2 \\ 3 & -2.0536 & 3 \\ 2 & 1/3 & -2.0536 \end{bmatrix} \times \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = 0$$

Here we should add the equation $w_1 + w_2 + w_3 = 1$ to the above system and calculate $W^T = (0.1571, 0.5936, 0.2493)$ the final respond as w_i

Theorem 1. For a positive and inverse matrix like the paired comparison matrix, we can achieve the special vector from the following relation:

$$W = \lim \frac{A^k.e}{e^T.A^k.e} \qquad k \to \infty$$

first we calculate $A^k.e$ (so that for k=1) $e^T = (1,1,...,1)$ in it

$$A^k.e = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1i} \\ \sum_{j=1}^{n} a_{2i} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{bmatrix}$$

Now we calculate the result of the expression $e^T.A^k.e$

$$e^t.A^k.e = e^t.(A^k.e) = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \times \begin{bmatrix} \sum_{j=1}^{n} a_{1i} \\ \sum_{j=1}^{n} a_{2i} \\ \vdots \\ \sum_{j=1}^{n} a_{ni} \end{bmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$$

in which the following is formed from: matrix A is in the power of K. $\frac{A^k.e}{e^T.A^k.e}$ Then, we plus the rows with each other until we achieve the column vector, and finally we normalize the result vector.

The prominent K (matrix power A) raise the power of matrix A(k→∞) the amount of W is near and nearer to the limit amount. So the difference of A^{K-1}, A^K matrix is very trivial, and we stop the calculation by giving one example to make the subject clearer.

Example. If the paired comparison matrix for 4 elements be as follows, by using the above theorem, we calculate the weight of element:

$$A = \begin{bmatrix} 1 & 1/9 & 1/3 & 1/4 \\ 9 & 1 & 3 & 2 \\ 3 & 1/3 & 1 & 1/2 \\ 4 & 1/2 & 2 & 1 \end{bmatrix}$$

After achieving $W^1 = \frac{A^1.e}{e^T.A^1.e}$, we should follow the next steps.

First repeat:

The first repetition is to calculate the plus of the number of each A matrix row (until we achieve the column vector) so that we normalize the column vector that we have achieved.

$$\begin{bmatrix} 1.695 \\ 15 \\ 4.833 \\ 7.50 \end{bmatrix} \xrightarrow{normalize} W^1 = \begin{bmatrix} 0.05837 \\ 0.51675 \\ 0.16651 \\ 0.25837 \end{bmatrix}$$

relation $W^2 = \dfrac{A^2.e}{e^T.A^2.e}$ from the W^2.

Second repeat:
In second repeat we achieve

$$A = \begin{bmatrix} 4 & 0.4583 & 1.5 & 0.8889 \\ 35 & 4 & 13 & 7.75 \\ 11 & 1.25 & 4 & 2.4167 \\ 18.5 & 2.1111 & 6.8333 & 4 \end{bmatrix}$$

where $W^2 = (0.05867, 0.51196, 0.15994, 0.26943)$. So we will have: W^3, W^4, W^5 . Now without entering to the details of necessary calculation we show its final amount in the following:

$W^3 = (0.05882, 0.51259, 0.15958, 0.26943)$ Third repeat:

$W^4 = (0.05882, 0.51261, 0.15971, 0.26886)$ Forth repeat:

$W^5 = (0.05882, 0.51261, 0.15971, 0.26886)$ Fifth repeat:

As we see by raising the amount k to the fifth repeat, the amount of W will be nearer to the constant amount (until five number after decimal) and continuing the calculation in this idea is not important [5].

2.2 Goal programming

Consider the following problem:

$$\begin{aligned} \text{Max} \quad & (f_1(x), f_2(x), \ldots, f_k(x)) \\ \text{s.t.} \quad & x \in X. \end{aligned} \tag{1}$$

where $f_1, f_2, \ldots f_k$ are objective functions and X is a non-empty feasible region. The model (1) is called multiple-objective programming (MOP).Goal programming is now an important area of multiple-criteria (or objective) optimization. The idea of goal programming is to establish a goal level of achievement for each criterion. GP is ideal for criteria with respect to which target values of achievement are of significance. In goal programming method it is required that the decision maker sets goals for each objective that he/she wishes to attain. A preferred solution is then defined as the one which minimizes the objective from the set goals. Thus a simple GP formulation is as follows:

$$\begin{aligned} \text{Min} \quad & (h_1(n,p), h_2(n,p), \ldots, h_t(n,p)) \\ \text{s.t.} \quad & f_i(x)+n_i-p_i=b_i, \\ & x \in X, \\ & n_i p_i=0, \quad i=1,2,\ldots,k, \\ & n_i, p_i \geq 0, \quad i=1,2,\ldots,k. \end{aligned} \tag{2}$$

where b_i $(i=1,2,...,k)$ is the specified goals by the decision maker for the objective, n_i and p_i is respectively, the under-achievement and over-achievement of the i-th goals. $h_j(n,p)$ $(j=1,2,...,t)$ is linear function of the deviation variables called the achievement function. For each function of f_i $(i=1,2,...,k)$ consider one of the following restrictions:

$$\begin{aligned} & f_i(x)=b_i, \quad && \text{(I)} \\ & f_i(x) \leq b_i, \quad && \text{(II)} \\ & f_i(x) \geq b_i, \quad && \text{(III)} \end{aligned} \tag{3}$$

Imposing one of the above restrictions to (1) (only one of these inequality is added to the problems), may result in infeasibility to the problem (1).To avoid this difficulty, the deviation variables of n_i and p_i are added to (3). Hence, we will have:

$$f_i(x)+n_i-p_i=b_i, \quad i=1,2,...,k$$

Consider the relationship between the original goal form (i.e.,$\leq$,$\geq$ .or =) and the deviation variables. It should be clear that

1 To satisfy $f_i(x) \leq b_i$, we must minimize the positive deviation p_i

2 To satisfy $f_i(x) \geq b_i$, we must minimize the negative deviation n_i

3 To satisfy $f_i(x) = b_i$, we must minimize both n_i and p_i

2.3 Weight restrictions

Exists in two ways:

1 Cone Ratio

2 Assurance Region

Here we discuss Assurance Region in two kinds of weight bounds for μ's and V's (μ is a weight of output) and (V is a weight of input) that we name them homogeneous and non-homogeneous.

1 Homogeneous weight bound is as follows:

$$l_{1i} \leq \frac{v_i}{v_1} \leq u_{1i} \qquad i=2,...,m$$

$$l_{1r} \leq \frac{\mu_r}{\mu_1} \leq u_{1r} \qquad r=2,...,s$$

2 Non-homogeneous weight bound is as follows:

$$\underline{v_i} \le v_i \le \overline{v_i} \qquad i = 1,2,...,m$$
$$\underline{\mu_r} \le \mu \le \overline{\mu_r} \qquad r = 1,2,...,s$$

Subject 1 and 2 are weight bounds that are easily transferable to the weight restrictions.

3 Evaluation of zero weights in weak efficient points and inefficient with AHP

In this part we describe the subject completely throughout the paper. We assume that we have n DMUs, each DMUj using m inputs (x_{ij} , i = 1, ...,m) and producing s outputs (y_{rj} , r = 1, ..., s); assume also that the relative efficiency of these DMUs is assessed with the CCR model of Charnes et al. [6].

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{s} \mu_r y_{ro} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, && j=1,...,n, \\ & v_i \ge 0, && i=1,...,m, \\ & \mu_r \ge 0, && r=1,...,s. \end{aligned} \tag{4}$$

Based on this DEA model we can partition the set of DMUs into the classes E, F, NE and NF [3]. The DMUs in E are pareto efficient; F is the set of weakly efficient unit. Finally, the DMUs in NE and NF are inefficient.

In this approach by the use of pareto efficient point and according to the decision-maker's idea and want, we find weights of DMUs separately, and within this approach after calculating the whole weights, we determine input-maximum and input-minimum among inputs, and determine output-maximum and output-minimum among outputs. Then we put them in the offered model in the following form.

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{s} \mu_r y_{ro} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, && j \notin E, \\ & \min_i \{w_i\} \le v_i \le \max_i \{w_i\}, && i=1,...,m, \\ & \min_r \{w_r\} \le \mu_r \le \max_r \{w_r\}, && r=1,...,s. \end{aligned} \tag{5}$$

In this model wi and wj are the weights of inputs and outputs respectively. We compute this weights with AHP method. And $\min_i\{wi\}$ and $\max_i\{wi\}$ are the upper bound and lower bound of the i-th input weight respectively, and $\min_r\{wr\}$ and $\max_r\{wr\}$ are the upper bound and lower bound of the r-th output weight respectively.

By the use of this model amount of efficiency change is less than primary amount. Obviously, these efficiency scores are lower than those provided by the CCR model as a result of eliminating the slacks, and consequently, for accounting all sources of inefficiency see examples1and 2. On the one hand, feasibility of the model depends on decision maker's idea and want,

But if the model is infeasible we can use the model expressed in a feasible interval for weights in data envelopment analysis [7]:

$$\begin{aligned}
\text{Max} \quad & Z_p=\sum_{r=1}^{s}\mu_r y_{rp}-M\left(\sum_{t=1}^{l_1}(P_t)+\sum_{h=1}^{l_2}(P'_h)\right) \\
\text{s.t.} \quad & \sum_{r=1}^{s}\mu_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij} \le 0, && j=1,\dots,n, \\
& \sum_{i=1}^{m}v_i x_{ip}=1, \\
& A_t\mu+n_t-p_t=\min\{w_r\}_r, && t=1,\dots,l_1,\quad r=1,\dots,s, \\
& A_t\mu+n_t-p_t=\max\{w_r\}_r, && t=1,\dots,l_1,\quad r=1,\dots,s, \\
& C_h V+n'_h-p'_h=\min\{w_i\}_i, && h=1,\dots,l_2,\quad i=1,\dots,m, \\
& C_h V+n'_h-p'_h=\max\{w_i\}_i, && h=1,\dots l_2,\quad i=1,\dots,m, \\
& \mu_r,v_i \ge 0, && r=1,\dots,s,\quad i=1,\dots,m, \\
& n_t,p_t,p'_t,n'_t \ge 0, && t=1,\dots,l_1,\quad h=1,2,\dots,l_2.
\end{aligned} \tag{6}$$

where $n_t, p_t; (t=1,\dots,l_1)$ and $n'_h, p'_h; (h=1,\dots,l_2)$ are deviation variable corresponding to weight restriction, and M is a very large positive number.

4 Example

In the following part, we present two examples of Lingo programmer, and this shows us that the solved examples are feasible.

The first example deals with weak efficient point and the second one deals with an inefficient point.

Example 1. Here we have three DMUs including two inputs and one output that have been solved the question with CCR model and the following results have been appeared.

Table 1 The result of the efficiency of the 3 DMUs

DMU_j	x_1	x_2	y	CCR	v_1	v_2	μ
1	1	2	1	1	+	+	+
2	2	1	1	1	+	+	+
3	1	5	1	1	1	0	1

(+: Sign shows calculated weights are positive)

Weights are appointed based on decision maker's views and wants, and weights are calculated by the use of paired comparison matrix and the presented approach, that weights for inputs and outputs of the three DMUs are the same as the following.
Weights: w(I) = [0.055, 0.39, 0.09], W(O) = [0.16, 0.6, 0.25].
Now, we solve the question by the use of weights and (2-2) model and the results are:

Table 2 The result of the DMU_ 3

		Score	v_1	v_2	*M*
DMU_3	*CCR*	1	1	0	1
	CCR_{new}	0.6	0.33	0.13	0.6

Example 2. Here we have four DMUs that include two inputs and one output and are solved like the first example.

Table 3 The result of the efficiency of the 4 DMUs

DMU_j	x_1	x_2	y	CCR	v_1	v_2	μ
1	1	8	1	1	+	+	+
2	2	3	1	1	+	+	+
3	2	5	1	1	+	+	+
4	10	2.5	1	0.8	0.04	0	0.8

Weights: w(I) = [0.055, 0.39, 0.09, 0.2], W(O) = [0.16, 0.6, 0.25, 0.3].
In the new model we have

Table 4 The result of the DMU_4

		Score	v_1	v_2	μ
DMU_4	*CCR*	0.8	0.4	0	0.8
	CCR_{new}	0.6	0.55	0.18	0.6

5 Conclusion

The presented approach in this paper in fact is a model that appoints weights based on decision maker's view and wants, and makes a positive weight until drawing inefficient and weak efficient points on efficient frontier.

This model appoints weights based on analytical hierarchy process and cannot guarantee that the model is feasible. In fact, on the one hand taking a decision with multiple criteria is not simple, and because of lack of standards, speed and accuracy of taking a decision is greatly decreased and this leads to the fact that taking a decision greatly depends on a person who takes the decision.

When there are a lot of DMUs, we can use the Expert choice (EC) software for calculation weights

Suggestions:

1 We suggest doing the work except of desire and idea of the decision maker for evaluation the weight of zero to interfere with inefficient and weak efficient point.
2 We change the weight bound in a way that we have only one pareto efficient point.
3 Trying the using of the decision maker's opinion and weight common, we calculate the efficiency of inefficient and weak efficient point.

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