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Two-stage DEA with Fuzzy Data

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Abstract Data envelopment analysis is a nonparametric technique checking efficiency of DMUs using math programming. In conventional DEA, it has been assumed that the status of each measure is clearly known as either input or output. Kao and Hwang [1] developed a data envelopment analysis (DEA) approach for measuring efficiency of decision processes which can be divided into two stages. The first stage uses inputs to generate outputs which become the inputs to the second stage. The first stage outputs are referred to as intermediate measures. The second stage then uses these intermediate measures to produce outputs. The data are crisp in the standard DEA model whereas there are many problems in the real life in which data may be uncertain. Thus, in this paper, a fuzzy version of two-stage DEA model with a symmetrical triangular fuzzy number is presented. The basic idea is to transform the fuzzy model into crisp linear programming by using $\alpha - cut$ approach. Finally, a numerical example is proposed to display the application of this method.

Keywords: Data Envelopment Analysis (DEA), Fuzzy Data, Two-stage DEA.

1 Introduction

DEA is a powerful tool in estimating efficiency of decision making units with multiple inputs and outputs. Charnes, *et al*., [2] were the pioneers of the field that introduced their first model named "CCR" in 1978. The assumption is that all the data have specific numerical values. Fuzzy DEA models can represent real world problems more realistically than the conventional DEA models. Several methods have been offered for solving the fuzzy CCR model. We can consider two approaches for solving fuzzy DEA. The first one defuzzifies the fuzzy model and changes it into the equivalent crisp model and the second one uses $\alpha - cuts$ to create interval valued linear programming that solves the fuzzy DEA by parametric programming.

Ghelej Beigi, Gholami [3] have proposed a model to estimate the efficiency score of DMUs with two-stage structure and fuzzy data. Then they suggested a new method to allocate resources to the DMUs. Their aim was preserving the efficiency score of DMUs after allocation.

Chen et al. [4] modeled the overall efficiency of a two-stage process as a weighted sum of the efficiencies of the two individual stages. Their method can be applied under both constant returns to scale (CRS) and variable returns to scale (VRS) assumptions. Kao and Hwang [1] developed a two-stage DEA modeling that considered the series relationship of the two sub-

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processes within the whole process. The efficiency of a DMU was decomposed into efficiencies of the two sub-DMUs through their framework. They consider a set of Taiwanese non-life insurance companies with a two-stage process of premium acquisition and profit generation. Tavana, Khalili-Damghani [5] proposed an efficient two-stage fuzzy DEA model to calculate the efficiency scores for a DMU and its sub-DMUs, they used the Stackelberg (leader–follower) game theory approach to prioritize and sequentially decompose the efficiency score of the DMU into a set of efficiency scores for its sub-DMUs. Their proposed models are linear and independent of the $\alpha - cut$ variables.

In this article, the two-stage DEA are considered which all the data of the DMUs are fuzzy with symmetrical triangular membership function. By using different $\alpha - cuts$ the fuzzy model convert to intervals [L, U], so we have interval linear programming. By applying S.SAATI M. [6] method a variable is defined which change the ILP problem to linear programming problem.

2 Two-Stage Model with Fuzzy Data

Suppose that, there are n two-stage structures DMUs to be evaluated, and that each $DMU_j, (j = 1,2,\dots,n)$ has m inputs to the first stage, x_{ij} $(i = 1,2,\dots,m)$, and D outputs from this stage $z_{dj}, (d = 1,2,\dots,D)$. These D outputs then become the inputs to the second stage, and are referred to as intermediate measures. The outputs from the second stage are denoted $y_{rj}, (r = 1,2,\dots,s)$.

Kao and Huang [1] assume that their model for measuring the overall efficiency of a DMU is given by:

$$
\begin{aligned}
\theta_o = Max\ & \sum_{d=1}^{D} n_d z_{do} + \sum_{r=1}^{s} u_r y_{ro} \\
s.t.\ & \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} n_d\, z_{do} = 1 \\
& \sum_{d=1}^{D} n_d z_{dj} - \sum_{i=1}^{m} v_i\, x_{ij} \le 0 \qquad (1) \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} n_d z_{dj} \le 0 \\
& n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n
\end{aligned}
$$

The following model determines the first stages efficiency(θ_o^{1*}), while maintaining the overall efficiency score at θ_o calculated from model (1)

$$
\begin{aligned}
\theta_o^{1*} = Max\ & \sum_{d=1}^{D} n_d z_{do} \\
s.t.\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{d=1}^{D} n_d z_{dj} - \sum_{i=1}^{m} v_i\, x_{ij} \le 0 \qquad (2) \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} n_d z_{dj} \le 0
\end{aligned}
$$

$$\sum_{r=1}^{s} u_r y_{ro} - \theta_o \sum_{i=1}^{m} v_i x_{io} + (1-\theta_o) \sum_{d=1}^{D} n_d z_{do} = 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n$$

The efficiency for the second stage is then calculated as

$$\theta_o^2 = \frac{\theta_o - w_1^*.\theta_o^{1*}}{w_2^*}$$

Where $w_1^*$and w_2^* represent optimal weights obtained from following model

$$w_1 = \frac{\sum_{i=1}^{m} v_i x_{io}}{\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} n_d z_{do}}, \; w_2 = \frac{\sum_{d=1}^{D} n_d z_{do}}{\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} n_d z_{do}}$$

The following model determines the second stages efficiency(θ_o^{2*}), while maintaining the overall efficiency score at θ_o calculated from model (1),

$$\theta_o^{2*} = Max \sum_{r=1}^{s} u_r y_{ro}$$
$$s.t. \quad \sum_{d=1}^{D} n_d z_{do} = 1$$
$$\sum_{d=1}^{D} n_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \qquad (3)$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} n_d z_{dj} \leq 0$$
$$\sum_{r=1}^{s} u_r y_{ro} - \theta_o \sum_{i=1}^{m} v_i x_{io} + (1-\theta_o) \sum_{d=1}^{D} n_d z_{do} = 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n$$

And the efficiency for the first stage is calculated as

$$\theta_o^1 = \frac{\theta_o - w_2^*.\theta_o^{2*}}{w_1^*}$$

The model (1), (2), (3) with fuzzy data can be written as:

$$\theta_o = Max \sum_{d=1}^{D} n_d \tilde{z}_{do} + \sum_{r=1}^{s} u_r \tilde{y}_{ro}$$
$$s.t. \quad \sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} n_d \tilde{z}_{do} = \tilde{1}$$
$$\sum_{d=1}^{D} n_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 \qquad (4)$$
$$\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} n_d \tilde{z}_{dj} \leq 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n$$

First stage:

$$\theta_o^{1*} = Max \sum_{d=1}^{D} n_d \tilde{z}_{do}$$
$$s.t. \quad \sum_{i=1}^{m} v_i \tilde{x}_{io} = \tilde{1}$$
$$\sum_{d=1}^{D} n_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 \qquad (5)$$

$$\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} n_d \tilde{z}_{dj} \le 0$$
$$\sum_{r=1}^{s} u_r \tilde{y}_{ro} - \theta_o \sum_{i=1}^{m} v_i \tilde{x}_{io} + (1-\theta_o) \sum_{d=1}^{D} n_d \tilde{z}_{do} = 0$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

Second stage:

$$\theta_o^{2*} = Max \sum_{r=1}^{s} u_r \tilde{y}_{ro}$$
$$s.t. \quad \sum_{d=1}^{D} n_d \tilde{z}_{do} = 1$$
$$\sum_{d=1}^{D} n_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0 \qquad (6)$$
$$\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} n_d \tilde{z}_{dj} \le 0$$
$$\sum_{r=1}^{s} u_r \tilde{y}_{ro} - \theta_o \sum_{i=1}^{m} v_i \tilde{x}_{io} + (1-\theta_o) \sum_{d=1}^{D} n_d \tilde{z}_{do} = 0$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

Where, '~' indicates the fuzziness.

Among the various types of fuzzy numbers, triangular fuzzy numbers are of more importance. In the sequel, we consider the inputs and outputs of DMUs as triangular fuzzy numbers.

Let $\tilde{x}_{ij} = \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right)$ and $\tilde{z}_{dj} = \left(z_{dj}^L, z_{dj}^M, z_{dj}^U\right)$ and $\tilde{y}_{rj} = \left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right)$ .Therefore (4), (5), (6) can be written as follows:

$$\theta_o = Max \sum_{d=1}^{D} n_d (z_{do}^L, z_{do}^M, z_{do}^U) + \sum_{r=1}^{s} u_r (y_{ro}^L, y_{ro}^M, y_{ro}^U)$$
$$s.t. \quad \sum_{i=1}^{m} v_i (x_{io}^L, x_{io}^M, x_{io}^U) + \sum_{d=1}^{D} n_d (z_{do}^L, z_{do}^M, z_{do}^U) = 1$$
$$\sum_{d=1}^{D} n_d \left(z_{dj}^L, z_{dj}^M, z_{dj}^U\right) - \sum_{i=1}^{m} v_i \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right) \le 0 \qquad (7)$$
$$\sum_{r=1}^{s} u_r \left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right) - \sum_{d=1}^{D} n_d \left(z_{dj}^L, z_{dj}^M, z_{dj}^U\right) \le 0$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n.$$

$$\theta_o^{1*} = Max \sum_{d=1}^{D} n_d (z_{do}^L, z_{do}^M, z_{do}^U)$$
$$s.t. \quad \sum_{i=1}^{m} v_i (x_{io}^L, x_{io}^M, x_{io}^U) = 1$$
$$\sum_{d=1}^{D} n_d \left(z_{dj}^L, z_{dj}^M, z_{dj}^U\right) - \sum_{i=1}^{m} v_i \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right) \le 0 \qquad (8)$$
$$\sum_{r=1}^{s} u_r \left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right) - \sum_{d=1}^{D} n_d \left(z_{dj}^L, z_{dj}^M, z_{dj}^U\right) \le 0$$

$$\sum_{r=1}^{s} u_r(y_{ro}^L, y_{ro}^M, y_{ro}^U) - \theta_o \sum_{i=1}^{m} v_i(x_{io}^L, x_{io}^M, x_{io}^U) + (1-\theta_o)\sum_{d=1}^{D} n_d(z_{do}^L, z_{do}^M, z_{do}^U) = 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n.$$

$$\theta_o^{2*} = Max \sum_{r=1}^{s} u_r(y_{ro}^L, y_{ro}^M, y_{ro}^U)$$
$$s.t. \quad \sum_{d=1}^{D} n_d(z_{do}^L, z_{do}^M, z_{do}^U) = 1$$
$$\sum_{d=1}^{D} n_d(z_{dj}^L, z_{dj}^M, z_{dj}^U) - \sum_{i=1}^{m} v_i(x_{ij}^L, x_{ij}^M, x_{ij}^U) \leq 0 \qquad (9)$$
$$\sum_{r=1}^{s} u_r(y_{rj}^L, y_{rj}^M, y_{rj}^U) - \sum_{d=1}^{D} n_d(z_{dj}^L, z_{dj}^M, z_{dj}^U) \leq 0$$
$$\sum_{r=1}^{s} u_r(y_{ro}^L, y_{ro}^M, y_{ro}^U) - \theta_o \sum_{i=1}^{m} v_i(x_{io}^L, x_{io}^M, x_{io}^U) + (1-\theta_o)\sum_{d=1}^{D} n_d(z_{do}^L, z_{do}^M, z_{do}^U) = 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n$$

Models (7), (8), (9) are possibility linear programming. There are several methods to solve it. In most of these methods for solving the possibility programming problem using $\alpha - cut$, the intervals in both sides of the constraints are compared with each other.

We apply the concept of $\alpha - cut$ to solve (7), (8), and (9) .By introducing $\alpha - cuts$ of objective function and constraints the following models are obtained:

$$\theta_o = Max \sum_{d=1}^{D} n_d(\alpha z_{do}^M + (1-\alpha)z_{do}^L, \alpha z_{do}^M + (1-\alpha)z_{do}^U) + \sum_{r=1}^{s} u_r(\alpha y_{ro}^M + (1-\alpha)y_{ro}^L, \alpha y_{ro}^M + (1-\alpha)y_{ro}^U)$$
$$s.t. \quad \sum_{i=1}^{m} v_i(\alpha x_{io}^M + (1-\alpha)x_{io}^L, \alpha x_{io}^M + (1-\alpha)x_{io}^U) + \sum_{d=1}^{D} n_d(\alpha z_{do}^M + (1-\alpha)z_{do}^L, \alpha z_{do}^M + (1-\alpha)z_{do}^U) = 1$$
$$\sum_{d=1}^{D} n_d\left(\alpha z_{dj}^M + (1-\alpha)z_{dj}^L, \alpha z_{dj}^M + (1-\alpha)z_{dj}^U\right) - \sum_{i=1}^{m} v_i\left(\alpha x_{ij}^M + (1-\alpha)x_{ij}^L, \alpha x_{ij}^M + (1-\alpha)x_{ij}^U\right) \leq 0 \qquad (10)$$
$$\sum_{r=1}^{s} u_r\left(\alpha y_{rj}^M + (1-\alpha)y_{rj}^L, \alpha y_{rj}^M + (1-\alpha)y_{rj}^U\right) - \sum_{d=1}^{D} n_d\left(\alpha z_{dj}^M + (1-\alpha)z_{dj}^L, \alpha z_{dj}^M + (1-\alpha)z_{dj}^U\right) \leq 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n$$

, First stage:

$$\theta_o^{1*} = Max \sum_{d=1}^{D} n_d(\alpha z_{do}^M + (1-\alpha)z_{do}^L, \alpha z_{do}^M + (1-\alpha)z_{do}^U)$$
$$s.t. \quad \sum_{i=1}^{m} v_i(\alpha x_{io}^M + (1-\alpha)x_{io}^L, \alpha x_{io}^M + (1-\alpha)x_{io}^U) = 1$$
$$\sum_{d=1}^{D} n_d\left(\alpha z_{dj}^M + (1-\alpha)z_{dj}^L, \alpha z_{dj}^M + (1-\alpha)z_{dj}^U\right) - \sum_{i=1}^{m} v_i\left(\alpha x_{ij}^M + (1-\alpha)x_{ij}^L, \alpha x_{ij}^M + (1-\alpha)x_{ij}^U\right) \leq 0 \qquad (11)$$
$$\sum_{r=1}^{s} u_r\left(\alpha y_{rj}^M + (1-\alpha)y_{rj}^L, \alpha y_{rj}^M + (1-\alpha)y_{rj}^U\right) - \sum_{d=1}^{D} n_d\left(\alpha z_{dj}^M + (1-\alpha)z_{dj}^L, \alpha z_{dj}^M + (1-\alpha)z_{dj}^U\right) \leq 0$$

$$\sum_{r=1}^{s} u_r(\alpha y_{ro}^M + (1-\alpha)y_{ro}^L, \alpha y_{ro}^M + (1-\alpha)y_{ro}^U) - \theta_o \sum_{i=1}^{m} v_i(\alpha x_{io}^M + (1-\alpha)x_{io}^L, \alpha x_{io}^M + (1-\alpha)x_{io}^U)$$
$$+ (1-\theta_o)\sum_{d=1}^{D} n_d(\alpha z_{do}^M + (1-\alpha)z_{do}^L, \alpha z_{do}^M + (1-\alpha)z_{do}^U) = 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n$$

Second stage:

$$\theta_o^{2*} = Max \sum_{r=1}^{s} u_r(\alpha y_{ro}^M + (1-\alpha)y_{ro}^L, \alpha y_{ro}^M + (1-\alpha)y_{ro}^U)$$
$$s.t. \quad \sum_{d=1}^{D} n_d(\alpha z_{do}^M + (1-\alpha)z_{do}^L, \alpha z_{do}^M + (1-\alpha)z_{do}^U) = 1$$
$$\sum_{d=1}^{D} n_d\left(\alpha z_{dj}^M + (1-\alpha)z_{dj}^L, \alpha z_{dj}^M + (1-\alpha)z_{dj}^U\right) - \sum_{i=1}^{m} v_i\left(\alpha x_{ij}^M + (1-\alpha)x_{ij}^L, \alpha x_{ij}^M + (1-\alpha)x_{ij}^U\right)$$
$$\leq 0 \qquad (12)$$
$$\sum_{r=1}^{s} u_r\left(\alpha y_{rj}^M + (1-\alpha)y_{rj}^L, \alpha y_{rj}^M + (1-\alpha)y_{rj}^U\right) - \sum_{d=1}^{D} n_d\left(\alpha z_{dj}^M + (1-\alpha)z_{dj}^L, \alpha z_{dj}^M + (1-\alpha)z_{dj}^U\right) \leq 0$$
$$\sum_{r=1}^{s} u_r(\alpha y_{ro}^M + (1-\alpha)y_{ro}^L, \alpha y_{ro}^M + (1-\alpha)y_{ro}^U) - \theta_o \sum_{i=1}^{m} v_i(\alpha x_{io}^M + (1-\alpha)x_{io}^L, \alpha x_{io}^M + (1-\alpha)x_{io}^U)$$
$$+ (1-\theta_o)\sum_{d=1}^{D} n_d(\alpha z_{do}^M + (1-\alpha)z_{do}^L, \alpha z_{do}^M + (1-\alpha)z_{do}^U) = 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n$$

By considering

$\underline{x}_{ij} = \alpha x_{ij}^M + (1-\alpha)x_{ij}^L$, $\bar{x}_{ij} = \alpha x_{ij}^M + (1-\alpha)x_{ij}^U$, $i = 1,\dots,m$, $j = 1,\dots,n$

$\underline{y}_{rj} = \alpha y_{rj}^M + (1-\alpha)y_{rj}^L$, $\bar{y}_{rj} = \alpha y_{rj}^M + (1-\alpha)y_{rj}^U$, $r = 1,\dots,s$, $j = 1,\dots,n$

$\underline{z}_{dj} = \alpha z_{dj}^M + (1-\alpha)z_{dj}^L$, $\bar{z}_{dj} = \alpha z_{dj}^M + (1-\alpha)z_{dj}^U$, $d = 1,\dots,D$, $j = 1,\dots,n$

Models (10), (11), (12) can written as following ILP problems:

$$\theta_o = Max \sum_{d=1}^{D} n_d(\underline{z}_{do}, \bar{z}_{do}) + \sum_{r=1}^{s} u_r\left(\underline{y}_{ro}, \bar{y}_{ro}\right)$$
$$s.t. \quad \sum_{i=1}^{m} v_i(\underline{x}_{io}, \bar{x}_{io}) + \sum_{d=1}^{D} n_d\left(\underline{z}_{do}, \bar{z}_{do}\right) = 1$$
$$\sum_{d=1}^{D} n_d\left(\underline{z}_{dj}, \bar{z}_{dj}\right) + \sum_{i=1}^{m} v_i\left(-\bar{x}_{ij}, -\underline{x}_{ij}\right) \leq 0 \qquad (13)$$
$$\sum_{r=1}^{s} u_r\left(\underline{y}_{rj}, \bar{y}_{rj}\right) + \sum_{d=1}^{D} n_d(-\bar{z}_{dj}, -\underline{z}_{dj}) \leq 0$$
$$n_d, v_i, u_r \geq 0, \quad j = 1,2,\dots,n.$$

First stage:

$$\theta_o^{1*} = Max \sum_{d=1}^{D} n_d(\underline{z}_{do}, \bar{z}_{do})$$
$$s.t. \quad \sum_{i=1}^{m} v_i(\underline{x}_{io}, \bar{x}_{io}) = 1$$

$$\sum_{d=1}^{D} n_d\left(\underline{z}_{dj}, \overline{z}_{dj}\right) + \sum_{i=1}^{m} v_i\left(-\overline{x}_{ij}, -\underline{x}_{ij}\right) \le 0 \qquad (14)$$
$$\sum_{r=1}^{s} u_r\left(\underline{y}_{rj}, \overline{y}_{rj}\right) + \sum_{d=1}^{D} n_d\left(-\overline{z}_{dj}, -\underline{z}_{dj}\right) \le 0$$
$$\sum_{r=1}^{s} u_r\left(\underline{y}_{ro}, \overline{y}_{ro}\right) + \theta_o \sum_{i=1}^{m} v_i\left(-\overline{x}_{io}, -\underline{x}_{io}\right) + \theta_o \sum_{d=1}^{D} n_d\left(-\overline{z}_{do}, -\underline{z}_{do}\right) + \sum_{d=1}^{D} n_d\left(\underline{z}_{do}, \overline{z}_{do}\right) = 0$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

Second stage

$$\theta_o^{2*} = Max \sum_{r=1}^{s} u_r\left(\underline{y}_{ro}, \overline{y}_{ro}\right)$$
$$s.t. \quad \sum_{d=1}^{D} n_d\left(\underline{z}_{do}, \overline{z}_{do}\right) = 1$$
$$\sum_{d=1}^{D} n_d\left(\underline{z}_{dj}, \overline{z}_{dj}\right) + \sum_{i=1}^{m} v_i\left(-\overline{x}_{ij}, -\underline{x}_{ij}\right) \le 0 \qquad (15)$$
$$\sum_{r=1}^{s} u_r\left(\underline{y}_{rj}, \overline{y}_{rj}\right) + \sum_{d=1}^{D} n_d\left(-\overline{z}_{dj}, -\underline{z}_{dj}\right) \le 0$$
$$\sum_{r=1}^{s} u_r\left(\underline{y}_{ro}, \overline{y}_{ro}\right) + \theta_o \sum_{i=1}^{m} v_i\left(-\overline{x}_{io}, -\underline{x}_{io}\right) + \theta_o \sum_{d=1}^{D} n_d\left(-\overline{z}_{do}, -\underline{z}_{do}\right) + \sum_{d=1}^{D} n_d\left(\underline{z}_{do}, \overline{z}_{do}\right) = 0$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

In this section by applying S.SAATI M. [6] method for solving ILP problems we have the following problems:

$$\theta_o = Max \sum_{d=1}^{D} n_d \hat{z}_{do} + \sum_{r=1}^{s} u_r \hat{y}_{ro}$$
$$s.t. \quad \sum_{i=1}^{m} v_i \hat{x}_{io} + \sum_{d=1}^{D} n_d \hat{z}_{do} = 1$$
$$\sum_{d=1}^{D} n_d \hat{z}_{dj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \le 0 \qquad (16)$$
$$\sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{d=1}^{D} n_d \hat{z}_{dj} \le 0$$
$$\underline{x}_{ij} \le \hat{x}_{ij} \le \overline{x}_{ij}$$
$$\underline{y}_{rj} \le \hat{y}_{rj} \le \overline{y}_{rj}$$
$$\underline{z}_{dj} \le \hat{z}_{dj} \le \overline{z}_{dj}$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

First stage:

$$\theta_o^{1*} = Max \sum_{d=1}^{D} n_d \hat{z}_{do}$$
$$s.t. \quad \sum_{i=1}^{m} v_i \hat{x}_{io} = 1$$

$$\sum_{d=1}^{D} n_d \hat{z}_{dj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \le 0 \qquad (14)$$
$$\sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{d=1}^{D} n_d \hat{z}_{dj} \le 0$$
$$\sum_{r=1}^{s} u_r \hat{y}_{ro} - \theta_o \sum_{i=1}^{m} v_i \hat{x}_{io} + (1-\theta_o) \sum_{d=1}^{D} n_d \hat{z}_{do} = 0$$
$$\underline{x}_{ij} \le \hat{x}_{ij} \le \overline{x}_{ij}$$
$$\underline{y}_{rj} \le \hat{y}_{rj} \le \overline{y}_{rj}$$
$$\underline{z}_{dj} \le \hat{z}_{dj} \le \overline{z}_{dj}$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

Second stage

$$\theta_o^{2*} = Max \sum_{r=1}^{s} u_r \hat{y}_{ro}$$
$$s.t. \quad \sum_{d=1}^{D} n_d \hat{z}_{do} = 1$$
$$\sum_{d=1}^{D} n_d \hat{z}_{dj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \le 0 \qquad (15)$$
$$\sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{d=1}^{D} n_d \hat{z}_{dj} \le 0$$
$$\sum_{r=1}^{s} u_r \hat{y}_{ro} - \theta_o \sum_{i=1}^{m} v_i \hat{x}_{io} + (1-\theta_o) \sum_{d=1}^{D} n_d \hat{z}_{do} = 0$$
$$\underline{x}_{ij} \le \hat{x}_{ij} \le \overline{x}_{ij}$$
$$\underline{y}_{rj} \le \hat{y}_{rj} \le \overline{y}_{rj}$$
$$\underline{z}_{dj} \le \hat{z}_{dj} \le \overline{z}_{dj}$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

Models (13), (14), (15) are nonlinear programming problem. In order to linearize the models we apply the following substitutions:

$\acute{x}_{ij} = v_i \hat{x}_{ij}$, $\acute{z}_{dj} = n_d \hat{z}_{dj}$, $\acute{y}_{rj} = u_r \hat{y}_{rj}$

By these substitutions models (13), (14), (15) will become linear programming as follows:

$$\theta_o = Max \sum_{d=1}^{D} \acute{z}_{do} + \sum_{r=1}^{s} \acute{y}_{ro}$$
$$s.t. \quad \sum_{i=1}^{m} \acute{x}_{io} + \sum_{d=1}^{D} \acute{z}_{do} = 1$$
$$\sum_{d=1}^{D} \acute{z}_{dJ} - \sum_{i=1}^{m} \acute{x}_{iJ} \le 0 \qquad (16)$$
$$\sum_{r=1}^{s} \acute{y}_{rJ} - \sum_{d=1}^{D} \acute{z}_{dJ} \le 0$$
$$v_i \underline{x}_{ij} \le \acute{x}_{iJ} \le v_i \overline{x}_{ij}$$

$$u_r \underline{y}_{rj} \le y'_{rJ} \le u_r \overline{y}_{rj}$$
$$n_d \underline{z}_{dj} \le z'_{dJ} \le n_d \overline{z}_{dj}$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

First stage:

$$\theta_o^{1*} = Max \sum_{d=1}^{D} z'_{do}$$
$$s.t. \quad \sum_{i=1}^{m} x'_{io} = 1$$
$$\sum_{d=1}^{D} z'_{dJ} - \sum_{i=1}^{m} x'_{iJ} \le 0 \qquad (17)$$
$$\sum_{r=1}^{s} y'_{rJ} - \sum_{d=1}^{D} z'_{dJ} \le 0$$
$$\sum_{r=1}^{s} y'_{ro} - \theta_o \sum_{i=1}^{m} x'_{io} + (1-\theta_o) \sum_{d=1}^{D} z'_{do} = 0$$
$$v_i \underline{x}_{ij} \le x'_{iJ} \le v_i \overline{x}_{ij}$$
$$u_r \underline{y}_{rj} \le y'_{rJ} \le u_r \overline{y}_{rj}$$
$$n_d \underline{z}_{dj} \le z'_{dJ} \le n_d \overline{z}_{dj}$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

Second stage

$$\theta_o^{2*} = Max \sum_{r=1}^{s} y'_{ro}$$
$$s.t. \quad \sum_{d=1}^{D} z'_{do} = 1$$
$$\sum_{d=1}^{D} z'_{dJ} - \sum_{i=1}^{m} x'_{iJ} \le 0 \qquad (18)$$
$$\sum_{r=1}^{s} y'_{rJ} - \sum_{d=1}^{D} z'_{dJ} \le 0$$
$$\sum_{r=1}^{s} y'_{ro} - \theta_o \sum_{i=1}^{m} x'_{io} + (1-\theta_o) \sum_{d=1}^{D} z'_{do} = 0$$
$$v_i \underline{x}_{ij} \le x'_{iJ} \le v_i \overline{x}_{ij}$$
$$u_r \underline{y}_{rj} \le y'_{rJ} \le u_r \overline{y}_{rj}$$
$$n_d \underline{z}_{dj} \le z'_{dJ} \le n_d \overline{z}_{dj}$$
$$n_d, v_i, u_r \ge 0, \quad j = 1,2,\dots,n$$

3 Numerical example

Suppose that we have 10 DMUs with one input(X^L, X^M, X^U) , one intermediate measure (Z^L, Z^M, Z^U) and two outputs$\left(Y_i^L, Y_i^M, Y_i^U\right), (i = 1,2)$. Table 1 presents the data of DMUs. We use the proposed models (16), (17), (18) to obtain the efficiency scores of DMUs.

Table 1 The data of DMUs

DMU	(X^L, X^M, X^U)	(Z^L, Z^M, Z^U)	$\left(Y_1^L, Y_1^M, Y_1^U\right)$	$\left(Y_2^L, Y_2^M, Y_2^U\right)$
1	(2,4,6)	(1,2,3)	(3,4,6)	(1,4,6)
2	(3,5,7)	(3,7,9)	(1,4,7)	(3,8,9)
3	(2,7,10)	(1,5,9)	(3,7,9)	(12,14,16)
4	(4,7,9)	(11,13,15)	(5,9,14)	(13,16,19)
5	(4,6,8)	(5,8,10)	(12,13,19)	(14,16,21)
6	(2,7,9)	(3,8,9)	(8,10,18)	(13,16,22)
7	(4,11,14)	(4,7,9)	(4,5,19)	(5,8,13)
8	(3,7,15)	(2,7,8)	(13,14,17)	(15,18,19)
9	(2,4,8)	(2,5,7)	(9,11,18)	(17,19,21)
10	(3,5,9)	(3,6,9)	(11,15,19)	(2,7,12)

The results from models (16), (17), (18) are represented in Table 2. The first column reports the overall efficiency of DMUs at different $\alpha - cuts$. The second and third column of Table 2 reports the efficiency score of DMUs at different $\alpha - cuts$ for each stage upon models (17), (18). The optimal weights obtained from model (16) are represented at Table 3.

Table 2 efficiency score of DMUs at different □ − □□□□

DMU	θ_o					θ_o^{1*}					θ_o^{2*}				
	$\alpha = 0$	□ = □.□□	□ = □.□	□ = □.□□	□ = □	□ = □	□ = □.□□	□ = □.□	□ = □.□□	□ = □	□ = □	□ = □.□□	□ = □.□	□ = □.□□	□ = □
1	1	0.897	0.714	0.557	0.392	1	0.813	0.556	0.386	0.269	1	1	1	1	0.846
2	1	0.907	0.766	0.666	0.559	1	1	1	1	0.754	1	0.818	0.532	0.414	0.301
3	1	1	0.87	0.662	0.482	1	1	1	0.626	0.385	1	1	0.745	1	0.737
4	0.92	0.845	0.777	0.713	0.662	1	1	1	1	1	0.84	0.689	0.554	0.426	0.324
5	1	1	1	0.897	0.712	1	1	1	0.927	0.718	1	1	1	0.986	0.704
6	1	1	1	0.809	0.595	1	1	1	0.861	0.615	1	1	1	0.827	0.561
7	1	1	0.762	0.514	0.337	1	1	0.711	0.486	0.343	1	1	0.849	0.584	0.321
8	1	1	1	0.841	0.655	1	1	1	0.725	0.538	1	1	1	1	0.871
9	1	1	1	0.971	0.805	1	1	1	0.943	0.673	1	1	1	1	1
10	1	1	1	0.947	0.785	1	1	1	0.9	0.646	1	1	1	1	1

Table 3 The optimal weights

DMU	□□					□□				
	□ = □	□ = □.□□	□ = □.□	□ = □.□□	□ = □	□ = □	□ = □.□□	□ = □.□	□ = □.□□	□ = □
1	0.55	0.626	0.729	0.812	0.788	0.45	0.374	0.271	0.188	0.212
2	0.55	0.574	0.6	0.625	0.57	0.45	0.426	0.4	0.375	0.43
3	0.55	0.575	0.6	0.727	0.724	0.45	0.425	0.4	0.273	0.276
4	0.55	0.575	0.6	0.625	0.506	0.45	0.425	0.4	0.375	0.494
5	0.55	0.575	0.6	0.643	0.582	0.45	0.425	0.4	0.357	0.418
6	0.55	0.575	0.6	0.66	0.619	0.45	0.425	0.4	0.34	0.381
7	0.55	0.575	0.678	0.774	0.745	0.45	0.425	0.322	0.226	0.255
8	0.55	0.575	0.6	0.697	0.65	0.45	0.425	0.4	0.303	0.35
9	0.55	0.575	0.6	0.64	0.597	0.45	0.425	0.4	0.36	0.403
10	0.55	0.575	0.6	0.65	0.607	0.45	0.425	0.4	0.35	0.393

4 Conclusion

The conventional DEA models uses a set of inputs to produce a set of outputs. In Yao Chens [4] method measuring efficiency of decision processes can be divided into two stages. The first stage uses inputs to generate outputs which become the inputs to the second stage. The

first stage outputs are referred to as intermediate measures. The second stage then uses these intermediate measures to produce outputs. Their method develops an additive efficiency decomposition approach wherein the overall efficiency is expressed as a (weighted) sum of the efficiencies of the individual stages.

All these assumptions occur when all the inputs and outputs of the two-stage DEA are crisp data. In this paper we consider that all of the inputs and outputs of two-stage DEA are triangular fuzzy numbers. Using fuzzy data, the model is converted to a possibility programming problem. We use Saati and Memariani [6] method for converting this problem into a crisp linear programming based on □−□□□. In the Saati and memariani model they define suitable variables to solve. The substitutions of these variables make the model non-linear. By further suitable substitutions the model is linearized. By solving a linear programming for different □−□□□s acceptable solutions is achieved for possibility mathematical programming problems.

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