International Journal of Applied Operational Research
Vol. 1, No. 1, pp. 49-63, Summer 2011

Journal homepage: www.ijorlu.ir

An Improvement for Efficiency Interval: Efficient and Inefficient Frontiers

H. Azizi*, R. Jahed

Received: January 21, 2011 ; **Accepted:** May 2, 2011

Abstract The performances of decision-making units (DMUs) can be evaluated from two different points of view optimistic and pessimistic and accordingly, two different efficiencies can be calculated for each DMU: the best relative efficiency and the worst relative efficiency. In the conventional methods of data envelopment analysis (DEA), only the best relative efficiency is evaluated. It is argued here that both efficiencies must be considered simultaneously, and any approach that considers only one of them will be biased. In this paper, it is proposed that to integrate both efficiencies in the form of an interval evaluates the overall performance of each DMU. To this end, a virtual DMU is used called the ideal DMU. The new DEA models with upper and lower bounds on efficiency are called the bounded DEA models. A numerical example is presented to illustrate the application of the proposed DEA models.

Keywords Data Envelopment Analysis (DEA), Optimistic and Pessimistic Efficiencies, Bounded DEA Models, Efficiency Interval.

1 Introduction

It has been known that the performances of decision-making units (DMUs) can be measured from different points of view. Data envelopment analysis (DEA), developed by Charnes et al. [1], measures the performances of DMUs from the optimistic point of view. The corresponding efficiencies are referred to as the best relative efficiencies or optimistic efficiencies, which are restricted to be greater than or equal to one. If a DMU is evaluated to have the best relative efficiency of one, then it is said to be DEA efficient or optimistic efficient; otherwise, it is said to be optimistic non-efficient. Optimistic efficient DMUs are usually thought to perform better than optimistic non-efficient DMUs.

On the other hand, the performances of DMUs can also be measured from the pessimistic point of view. The efficiencies measured from the pessimistic viewpoint may be referred to as the worst relative efficiencies or pessimistic efficiencies, which are measured within the range of less than or equal to one. Contrary to the best relative efficiencies that determine an efficiency frontier, the worst relative efficiencies of DMUs define an inefficiency frontier. If a

*** Corresponding author.** (✉)
E-mail: hazizi@iaupmogan.ac.ir (H. Azizi)

H. Azizi
Parsabad Moghan Branch, Islamic Azad University, Parsabad Moghan, Iran

R. Jahed
Germi Branch, Islamic Azad University, Germi, Iran

DMU is evaluated to have the worst relative efficiency of one, then it is said to be pessimistic inefficient; otherwise, it is said to be pessimistic non-inefficient. Pessimistic inefficient DMUs are usually thought to perform worse than pessimistic non-inefficient DMUs.

From the above analyses we can see that efficiency is a relative measure. It can be measured either within the range of less than or equal to one, or within the range of greater than or equal to one. When measured within different ranges, it has different meanings. The resultant assessment conclusions are usually different. Any assessment using only one type of efficiency is obviously one-sided. Ideally, both types of efficiencies should be used at the same time to assess the performances of DMUs.

In order to have an overall assessment of the performance of each DMU, we must consider both optimistic and pessimistic efficiencies simultaneously. Entani et al. [2] studied the performances of DMUs from both optimistic and pessimistic points of view. In their DEA models, optimistic and pessimistic efficiencies are used to form an interval. Their idea was that the efficiency of a DMU is the interval between the optimistic and the pessimistic values. However, their DEA model for computation of the optimistic efficiency of each DMU has a major drawback; namely, it does not take into account some of the input and output data. Their method practically considers the data of only one input and one output for the DMU under evaluation and ignores the rest of the input and output data. Furthermore, their model is not able to identify DEA-efficient DMUs adequately.

Wang and Luo [3] measure the optimistic and the pessimistic efficiencies of DMUs by introducing two virtual DMUs: ideal DMU (IDMU) and anti-ideal DMU, and integrate the two efficiencies into a relative closeness index, which servers as the basis for ranking DMUs. But in most cases, their models use fixed weights for all DMUs.

Wang and Yang [4] proposed a bounded DEA models for precise data. The bounded DEA models makes the most of all input and output information to measure both the best and the worst possible relative efficiencies of each DMU by introducing a virtual anti-ideal DMU, which consumes the most inputs only to produce the least outputs. It can therefore identify both the efficiency and inefficiency frontiers.

In this paper, we reconsider the problem of performance measurement. We measure the efficiencies of DMUs within the range of an interval so that the worst and the best relative efficiencies can be measured within a unified DEA model framework. In order to determine the range of interval efficiency, a virtual IDMU is introduced, whose performance is definitely the best among all the DMUs. So, its worst relative efficiency can be utilized as the constraint on the lower bound efficiencies of DMUs. A new DEA model with the upper and lower bounds on efficiencies is thus developed to compute the worst and the best relative efficiencies of each DMU, which constitute an interval to give an overall assessment of the performance of each DMU.

The rest of this article is organized as follows. Section 2 introduces basic DEA models for measurement of optimistic and pessimistic efficiencies of DMUs. Section 3 initially discusses Entani et al.'s [2] models and then presents the bounded DEA models. Section 4 compares the bounded DEA models and Entani et al.'s [2] models using a numerical example. Conclusions are set forth in Section 5.

2 DEA models for measuring the best and the worst relative efficiencies

2.1 DEA model for measuring the best relative efficiencies of DMUs

Assume that we want to evaluate n DMUs, each DMU consuming different amounts of m

inputs to produce r different outputs. In other words, DMU$_j$ ($j = 1,\ldots,n$) consumes the amounts x_{ij} ($i = 1,\ldots,m$) of inputs and produces the amounts y_{rj} ($r = 1,\ldots,s$) of outputs, all of which are known and non-negative, and each DMU has at least one positive input and one positive output.

In order to measure the efficiency of DMU$_j$ relative to other DMUs, Charnes et al. [1] developed the following CCR model, which measures the best relative efficiency of DMUs in the output-oriented mode:

$$\begin{aligned} \text{Min} \quad & \theta_o = \frac{\sum_{i=1}^{m} v_i x_{io}}{\sum_{r=1}^{s} u_r y_{ro}} \\ \text{s.t.} \quad & \theta_j = \frac{\sum_{i=1}^{m} v_i x_{ij}}{\sum_{r=1}^{s} u_r y_{rj}} \geq 1, \quad j = 1,\ldots,n, \\ & u_r, v_i \geq \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{1}$$

Where DMU$_o$ denotes the DMU under evaluation; v_i ($i = 1,\ldots,m$) and u_r ($r = 1,\ldots,s$) are decision variables; and ε is the non-Archimedean infinitesimal. Using Charnes and Cooper's [5] transformation, the above fractional programming model can be converted into the following linear programming (LP) model:

$$\begin{aligned} \text{Min} \quad & \theta_o = \sum_{i=1}^{m} v_i x_{io} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \geq 0, \quad j = 1,\ldots,n, \\ & \sum_{r=1}^{s} u_r y_{ro} = 1, \\ & u_r, v_i \geq \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{2}$$

If there exists a set of positive weights that makes $\theta_o^* = 1$, then DMU$_o$ is referred to be DEA efficient; otherwise, we call it to be DEA non-efficient rather than DEA inefficient because DEA non-efficient does not necessarily mean DEA inefficient. In fact, DEA efficient and DEA inefficient are only two extreme cases. For n different DMUs, there is a total number of n LP models to be solved. Accordingly, there are n different sets of weights, which are the basis to calculate the cross-efficiency matrix [6].

2.2 DEA model for measuring the worst relative efficiencies of DMUs

Efficiency is a relative measure and can be measured within different ranges. The CCR model measures the optimistic efficiency of each DMU by minimization within the range of greater than or equal to one. If the efficiency of a DMU is measured by maximization within the range of less than or equal to one, then we have the so-called pessimistic efficiency or the worst relative efficiency. The pessimistic efficiency of DMU_o can be measured by the following pessimistic DEA model [7, 8]:

$$\begin{aligned} \text{Max} \quad & \varphi_o = \frac{\sum_{i=1}^{m} v_i x_{io}}{\sum_{r=1}^{s} u_r y_{ro}} \\ \text{s.t.} \quad & \varphi_j = \frac{\sum_{i=1}^{m} v_i x_{ij}}{\sum_{r=1}^{s} u_r y_{rj}} \le 1, \quad j = 1, \dots, n, \\ & u_r, v_i \ge \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned} \tag{3}$$

Which can be further transformed into the following equivalent LP model:

$$\begin{aligned} \text{Max} \quad & \varphi_o = \sum_{i=1}^{m} v_i x_{io} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \le 0, \quad j = 1, \dots, n, \\ & \sum_{r=1}^{s} u_r y_{ro} = 1, \\ & u_r, v_i \ge \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned} \tag{4}$$

Efficiencies determined by the above LP model (4) are referred to as the worst relative efficiencies. Contrary to the CCR model (2) that determines an efficiency frontier for n DMUs, model (4) determines an inefficiency frontier for them. We refer to those DMUs lying on the inefficiency frontier to be DEA inefficient, while those not lying on the inefficiency frontier to be DEA non-inefficient.

Since the best relative efficiencies measure the best performances of DMUs, while the worst relative efficiencies measure their worst performances, such two types of relative efficiencies usually lead to two distinctive assessment conclusions. Any assessment using only one type of efficiency is obviously not all-sided. Therefore, there is a clear need to combine both types of relative efficiencies and give an overall measurement and assessment of the performance of each DMU.

3 Bounded DEA models for measuring interval efficiencies of DMUs
3.1 Review of existing work

Since the worst and the best relative efficiencies are measured within different ranges, they are incomparable. Therefore, they cannot be directly used to form an efficiency interval for each DMU. In order to be able to generate an interval efficiency assessment for each DMU, Entani et al. [2] constructed the following upper and lower bounds mathematical programming model for DMU_o:

$$\begin{aligned} \text{Max/Min} \quad & \theta = \frac{\sum_{i=1}^{m} v_i x_{io} / \sum_{r=1}^{s} u_r y_{ro}}{\max_j \left\{ \sum_{i=1}^{m} v_i x_{ij} / \sum_{r=1}^{s} u_r y_{rj} \right\}} \\ \text{s.t.} \quad & u_r, v_i \geq 0, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{5}$$

Where the upper bound model was further transformed into the model below, which is equivalent to the standard model (3) and can be solved through model (4):

$$\begin{aligned} \text{Max} \quad & \theta_o^U = \sum_{i=1}^{m} v_i x_{io} \Big/ \sum_{r=1}^{s} u_r y_{ro} \\ \text{s.t.} \quad & \max_j \left\{ \sum_{i=1}^{m} v_i x_{ij} \Big/ \sum_{r=1}^{s} u_r y_{rj} \right\} = 1, \\ & u_r, v_i \geq 0, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{6}$$

While the lower bound model was converted into the following model, which cannot be replaced with an equivalent LP problem:

$$\begin{aligned} \text{Min} \quad & \theta_o^L = \sum_{i=1}^{m} v_i x_{io} \Big/ \sum_{r=1}^{s} u_r y_{ro} \\ \text{s.t.} \quad & \max_j \left\{ \sum_{i=1}^{m} v_i x_{ij} \Big/ \sum_{r=1}^{s} u_r y_{rj} \right\} = 1, \\ & u_r, v_i \geq 0, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{7}$$

By assuming that $\sum_{i=1}^{m} v_i x_{ij} / \sum_{r=1}^{s} u_r y_{rj} = 1$ for each DEA inefficient unit (pessimistic inefficient DMU), Entani et al. [2] divided the above model (7) into the following ℓ sub-optimization problems ($J_1,\ldots,J_\ell$) where ℓ is the number of pessimistic inefficient DMUs and $J_1,\ldots,J_\ell$ are the DMUs which are pessimistic inefficient:

$$\begin{aligned} \operatorname{Min} \quad & \theta_{oj}^{L} = \sum_{i=1}^{m} v_i x_{io} \Big/ \sum_{r=1}^{s} u_r y_{ro} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} \Big/ \sum_{r=1}^{s} u_r y_{rj} = 1, \\ & u_r, v_i \geq 0, \quad r = 1, \ldots, s, \quad i = 1, \ldots, m. \end{aligned} \tag{8}$$

Which can be further simplified as the ℓ LP models below:

$$\begin{aligned} \operatorname{Min} \quad & \theta_{oj}^{L} = \sum_{i=1}^{m} v_i x_{io} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{ro} = 1, \\ & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} = 0, \\ & u_r, v_i \geq 0, \quad r = 1, \ldots, s, \quad i = 1, \ldots, m. \end{aligned} \tag{9}$$

Let θ_{oj}^{L*} be the optimum objective function value of the above LP model (9). It is obvious that when $j = o$, $\theta_{oj}^{L*} = 1$. So, the lower bound efficiency of DMU_o was finally determined by

$$\theta_{o}^{L*} = 1 \wedge \underset{j \neq o}{\operatorname{Min}} \{\theta_{oj}^{L*}\} \tag{10}$$

where $a \wedge b = \min\{a, b\}$. Accordingly, the efficiency interval for DMU_o is denoted as $[\theta_o^{L*}, \theta_o^{U*}]$, where θ_o^{U*} is the optimum objective function value of the upper bound model (6). Carefully analyzing models (7)-(9), the following drawbacks have been found:

1. One important feature of measuring the best relative efficiencies of DMUs is to identify DEA efficient DMUs, which perform the best among all DMUs from the optimistic point of view, and to determine an efficiency frontier so that the decision maker or assessor knows which DMUs are DEA efficient and which DMUs are not. But models (7)-(9) fail to do so. They can identify only one DMU with the smallest lower bound efficiency and not all DEA efficient DMUs. Accordingly, they cannot determine the efficiency frontier. So, much information useful to the decision maker or assessor was lost.
2. Models (8)-(9) use only one DMU, i.e. DMU_j as the reference set to compute the lower bound efficiency of DMU_o. So, model (9) has only two constraint conditions, which leads to only one input and one output weights to be nonzero and all the other input and output weights to be zero. That is to say, only one input and one output data of DMU_o were effectively used and all the other input and output data were ignored when computing its lower bound efficiency. This is obviously unreasonable and unacceptable.

Evidently, the model (5) cannot reasonably measure the best relative efficiencies of

DMUs and cannot determine the efficiency frontier. So, in what follows, we will develop a new DEA model with the constraint of the upper and lower bounds on efficiency. For convenience and simplicity, we refer to it as a bounded DEA model. The Bounded DEA model measures the performances of DMUs within the range of an interval and thus can effectively make the most of all the input and output data to measure both the best and the worst relative efficiencies of DMUs.

3.2 Bounded DEA models for crisp data

In order to reasonably measure the interval efficiencies of DMUs, we first introduce the concept of IDMU.

Definition 1. An IDMU is a virtual DMU, which can use the least inputs to generate the most outputs.

According to the above definition, we denote by $x_i^{\min}$ ($i = 1,\ldots,m$) and $y_r^{\max}$ ($r = 1,\ldots,s$) the inputs and outputs of the IDMU, respectively, where $x_i^{\min}$ is the minimum of the *i*-th input and $y_r^{\max}$ the maximum of the *r*-th output. They are determined by the following formulae:

$$x_i^{\min} = \underset{j}{\mathrm{Min}}\{x_{ij}\}, \quad i = 1,\ldots,m,$$

$$y_r^{\max} = \underset{j}{\mathrm{Max}}\{y_{rj}\}, \quad r = 1,\ldots,s.$$

Since the IDMU utilizes the least inputs to produce the most outputs, its performance is without doubt the best among all the DMUs. So, its efficiency should be the smallest at any circumstance.

Let φ_{IDMU}^* be the worst relative efficiency of the IDMU. Then it can be determined by using the following fractional programming model [8]:

$$\begin{aligned} \text{Max} \quad & \varphi_{\text{IDMU}} = \frac{\sum_{i=1}^{m} v_i x_i^{\min}}{\sum_{r=1}^{s} u_r y_r^{\max}} \\ \text{s.t.} \quad & \varphi_j = \frac{\sum_{i=1}^{m} v_i x_{ij}}{\sum_{r=1}^{s} u_r y_{rj}} \le 1, \quad j = 1,\ldots,n, \\ & u_r, v_i \ge \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{11}$$

which can be solved through the following LP model:

$$\begin{aligned}
\text{Max} \quad & \varphi_{IDMU} = \sum_{i=1}^{m} v_i x_i^{min} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \le 0, \quad j = 1,\ldots,n, \\
& \sum_{r=1}^{s} u_r y_r^{max} = 1, \\
& u_r, v_i \ge \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m.
\end{aligned} \tag{12}$$

Where ε is the non-Archimedean infinitesimal. After φ^*_{IDMU} is determined, we know that the efficiencies of all the DMUs cannot be less than it. Therefore, we can measure the efficiencies of DMUs within the range of interval $[\varphi^*_{IDMU}, 1]$. The following pair of fractional programming models reflects this idea [4]:

$$\begin{aligned}
\text{Max / Min} \quad & \phi_o = \frac{\sum_{i=1}^{m} v_i x_{io}}{\sum_{r=1}^{s} u_r y_{ro}} \\
\text{s.t.} \quad & \varphi^*_{IDMU} \le \frac{\sum_{i=1}^{m} v_i x_{ij}}{\sum_{r=1}^{s} u_r y_{rj}} \le 1, \quad j = 1,\ldots,n, \\
& u_r, v_i \ge \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m.
\end{aligned} \tag{13}$$

which can be equivalently transformed into the following pair of LP models:

$$\begin{aligned}
\text{Max / Min} \quad & \phi_o = \sum_{i=1}^{m} v_i x_{io} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \le 0, \quad j = 1,\ldots,n, \\
& \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r (\varphi^*_{IDMU} y_{rj}) \ge 0, \quad j = 1,\ldots,n, \\
& \sum_{r=1}^{s} u_r y_{ro} = 1, \\
& u_r, v_i \ge \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m.
\end{aligned} \tag{14}$$

Both models (13) and (14) are called bounded DEA models. Let ϕ_o^{U*} and ϕ_o^{L*} be the maximum and the minimum of the above objective function, respectively. Then, they form an efficiency interval, denoted by $[\phi_o^{L*}, \phi_o^{U*}]$, which measures the worst and the best relative efficiencies of DMU_o and its efficiency range. Repeating the above solution process for each DMU, we can obtain both the worst and the best relative efficiencies of all the DMUs and their efficiency intervals $[\phi_j^{L*}, \phi_j^{U*}]$ ($j = 1,\ldots,n$).

About the interval efficiency, $[\phi_o^{L*}, \phi_o^{U*}]$, we have the following definitions:

Definition 2. DMU_o is said to be DEA inefficient if and only if $\phi_o^{U*} = 1$, otherwise it is said to be DEA non-inefficient.

Definition 3. DMU_o is said to be DEA efficient if and only if $\phi_o^{L*} = \varphi_{IDMU}^*$, otherwise it is said to be DEA non-efficient.

Definition 4. DMU_o is said to be DEA unspecified if and only if it is neither DEA efficient nor DEA inefficient.

Definition 5. DMU_o is said to be DEA peculiar if and only if it is both DEA efficient and DEA inefficient.

All the DEA efficient DMUs determine an efficient production frontier, while all the DEA inefficient DMUs together define an inefficient production frontier called the inefficiency frontier. For those DEA unspecified units, they are always enveloped by both the efficiency and the inefficiency frontiers. Note that some DMU(s) may be both DEA inefficient and DEA efficient. Such DMUs have the widest efficiency interval $[\varphi_{IDMU}^*, 1]$. Their evaluations in fact contain the biggest uncertainty [8].

3.3 Bounded DEA models with preference information on weights

Traditional DEA approach often uses so-called assurance region approach or cone-ratio method to restrict factor weights u_r ($r = 1,\ldots,s$) and/or v_i ($i = 1,\ldots,m$). As a matter of fact, these two approaches are also applicable to the bounded DEA models (13) and (14). Here we consider how to incorporate the decision maker or the assessor's preference information on input and output weights into the bounded DEA models [4].

Since u_r ($r = 1,\ldots,s$) and v_i ($i = 1,\ldots,m$) are factor weights with different dimensions, they are usually incomparable. To take into account the decision maker or the assessor's preference information, we first carry out scale transformation to eliminate the dimension for each output and input factor.

Let

$$\tilde{y}_{rj} = \frac{y_{rj}}{\max_j \{y_{rj}\}} = \frac{y_{rj}}{y_r^{\max}}, \quad r = 1,\ldots,s; \quad j = 1,\ldots,n, \tag{15}$$

$$\tilde{x}_{ij} = \frac{x_{ij}}{\max_j \{x_{ij}\}} = \frac{x_{ij}}{x_i^{\max}}, \quad i = 1,\ldots,m; \quad j = 1,\ldots,n. \tag{16}$$

The scale-transformed input and output data are of no dimensions and are all within the range of $[0, 1]$. Since DEA model has the property of unit-invariance, the use of scale transformation to input and output data does not change the efficiencies of DMUs. Therefore, we have

$$\theta_o = \frac{\sum_{i=1}^{m} v_i x_{io}}{\sum_{r=1}^{s} u_r y_{ro}} = \frac{\sum_{i=1}^{m} \tilde{v}_i \tilde{x}_{io}}{\sum_{r=1}^{s} \tilde{u}_r \tilde{y}_{ro}} \tag{17}$$

where $\tilde{u}_r$ ($r = 1,\dots,s$) and $\tilde{v}_i$ ($i = 1,\dots,m$) are the factor weights corresponding to the scale-transformed output and input data. They have no dimensions and are thus comparable. They can be utilized to express the decision maker or assessor's preference on outputs and inputs. According to the relative importance between outputs and inputs, the decision maker or assessor may provide various types of preference information on outputs and inputs such as $\tilde{u}_{r_1} \geq \tilde{u}_{r_2}$, $\tilde{v}_{i_1} \geq \tilde{v}_{i_2}$, $\tilde{u}_{r_3} = \tilde{u}_{r_4}$, $\tilde{v}_{i_3} = \tilde{v}_{i_4}$, $\alpha \leq \tilde{u}_{r_5} / \tilde{u}_{r_6} \leq \beta$, $\gamma \leq \tilde{v}_{i_5} / \tilde{v}_{i_6} \leq \delta$, and so on. Substituting (15) and (16) into (17), we have

$$\theta_o = \frac{\sum_{i=1}^{m} v_i x_{io}}{\sum_{r=1}^{s} u_r y_{ro}} = \frac{\sum_{i=1}^{m} \tilde{v}_i \tilde{x}_{io}}{\sum_{r=1}^{s} \tilde{u}_r \tilde{y}_{ro}} = \frac{\sum_{i=1}^{m} (\tilde{v}_i / x_i^{\max}) x_{io}}{\sum_{r=1}^{s} (\tilde{u}_r / y_r^{\max}) y_{ro}} \tag{18}$$

from which we know that

$$\tilde{u}_r = u_r y_r^{\max}, \quad r = 1,\dots,s, \tag{19}$$

$$\tilde{v}_i = v_i x_i^{\max}, \quad i = 1,\dots,m. \tag{20}$$

These are two very important formulae, which show that the factor weights u_r ($r = 1,\dots,s$) and v_i ($i = 1,\dots,m$) multiplied by the maxima of output and input data can be used to express the decision maker or the assessor's preference. For example, the decision maker preference information mentioned above can be equivalently expressed as

$$u_{r_1} y_{r_1}^{\max} \geq u_{r_2} y_{r_2}^{\max},\ v_{i_1} x_{i_1}^{\max} \geq v_{i_2} x_{i_2}^{\max},\ u_{r_3} y_{r_3}^{\max} = u_{r_4} y_{r_4}^{\max},\ v_{i_3} x_{i_3}^{\max} = v_{i_4} x_{i_4}^{\max},$$
$$\alpha \leq u_{r_5} y_{r_5}^{\max} / u_{r_6} y_{r_6}^{\max} \leq \beta,\ \gamma \leq v_{i_5} x_{i_5}^{\max} / v_{i_6} x_{i_6}^{\max} \leq \delta.$$

Such preference information on factor weights can be easily incorporated into the bounded DEA models.

Let

$$\mathscr{U} = \{u = (u_r) \mid u_{r_1} y_{r_1}^{\max} \geq u_{r_2} y_{r_2}^{\max}, u_{r_3} y_{r_3}^{\max} = u_{r_4} y_{r_4}^{\max}, \alpha \leq u_{r_5} y_{r_5}^{\max} / u_{r_6} y_{r_6}^{\max} \leq \beta\} \tag{21}$$

$$\mathscr{V} = \{v = (v_i) \mid v_{i_1} x_{i_1}^{\max} \geq v_{i_2} x_{i_2}^{\max}, v_{i_3} x_{i_3}^{\max} = v_{i_4} x_{i_4}^{\max}, \gamma \leq v_{i_5} x_{i_5}^{\max} / v_{i_6} x_{i_6}^{\max} \leq \delta\} \tag{22}$$

Then the bounded DEA models with the preference information on weights can be expressed as follows:

$$
\begin{aligned}
\text{Max / Min} \quad & \phi_o = \sum_{i=1}^{m} v_i x_{io} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \le 0, \quad j = 1,\dots,n, \\
& \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r (\varphi_{IDMU}^{*} y_{rj}) \ge 0, \quad j = 1,...,n, \\
& \sum_{r=1}^{s} u_r y_{ro} = 1, \\
& (u_r) \in \mathscr{U}, \\
& (v_i) \in \mathscr{V}, \\
& u_r, v_i \ge \varepsilon, \quad r = 1,\dots,s, \quad i = 1,\dots,m.
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
\text{Max} \quad & \varphi_{IDMU} = \sum_{i=1}^{m} v_i x_i^{min} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \le 0, \quad j = 1,\dots,n, \\
& \sum_{r=1}^{s} u_r y_r^{max} = 1, \\
& (u_r) \in \mathscr{U}, \\
& (v_i) \in \mathscr{V}, \\
& u_r, v_i \ge \varepsilon, \quad r = 1,\dots,s, \quad i = 1,\dots,m.
\end{aligned} \tag{24}
$$

4 A numerical example

We now examine a numerical example using the bounded DEA model to illustrate its application in real-world performance measurement.

Consider a performance-measurement problem with ten DMUs, each DMU with one input and two outputs. The data set is taken from Entani et al. [2] and shown in Table 1, where all inputs are normalized to one for simplicity.

Table 1 Data for 10 DMUs with one input and two outputs

DMU	Input	Output 1	Output 2
A	1	1	8
B	1	2	3
C	1	2	6
D	1	3	3
E	1	3	7
F	1	4	2
G	1	4	5
H	1	5	2
I	1	6	2
J	1	7	1
IDMU	1	7	8

The best and the worst relative efficiencies of each DMU are calculated by using models (2) and (4), respectively, and the results are recorded in the second and third columns of Table 2. The non-Archimedean infinitesimal is set to be $\varepsilon = 10^{-10}$.

Table 2 Interval efficiencies and relative efficiencies of the 10 DMUs

DMU	Optimistic efficiency	Pessimistic efficiency	Interval efficiency	
			Bounded DEA models	Entani et al.'s DEA models
A	1.0000	1.0000	[0.3478, 1.0000]	[0.1250, 1.0000]
B	1.9167	1.0000	[0.6666, 1.0000]	[0.3333, 1.0000]
C	1.2143	0.8125	[0.4223, 0.8125]	[0.1667, 0.8125]
D	1.5333	0.8889	[0.5333, 0.8889]	[0.3333, 0.8889]
E	1.0000	0.5909	[0.3478, 0.5909]	[0.1429, 0.5909]
F	1.4375	1.0000	[0.5000, 1.0000]	[0.2500, 1.0000]
G	1.0455	0.5714	[0.3636, 0.5714]	[0.2000, 0.5714]
H	1.2105	0.9091	[0.4210, 0.9091]	[0.2000, 0.9091]
I	1.0455	0.8333	[0.3636, 0.8333]	[0.1667, 0.8333]
J	1.0000	1.0000	[0.3478, 1.0000]	[0.1429, 1.0000]

From the angle of the best relative efficiency, DMU_A, DMU_E and DMU_J are all evaluated to be DEA efficient. They together determine an efficiency frontier, which is shown in Fig.1. Their performances are usually thought to be better than any other DMUs that are evaluated to be DEA non-efficient. The performances of those DEA non-efficient DMUs are rated to be $DMU_B \succ DMU_D \succ DMU_F \succ DMU_C \succ DMU_H \succ DMU_I \sim DMU_G$, where the symbol '~' means 'be indifferent to', while the symbol '$\succ$' represents 'performs worse than'.

However, when the DMUs are evaluated from the viewpoint of the worst relative efficiencies, DMU_A, DMU_B, DMU_F and DMU_J are all evaluated to be DEA inefficient. They together define an inefficiency frontier, which is also shown in Fig.1. Their performances are usually thought to be worse than any other DMUs that are evaluated to be DEA non-inefficient. The performances of those DEA non-inefficient DMUs are rated to be $DMU_H \succ DMU_D \succ DMU_I \succ DMU_C \succ DMU_E \succ DMU_G$.

The above assessments are based on different points of view and may therefore be different. For example, for DMU_A and DMU_J, when they are evaluated from the optimistic point of view, they are evaluated to be optimistic efficient, which means they perform better than any other DMUs. However, when they are evaluated from the pessimistic point of view, they are both evaluated to be pessimistic inefficient, which means they perform worse than any other DMUs. Such two assessment results are obviously in conflict with each other. Any assessment conclusion considering only one point of view is apparently one-sided, unrealistic, and unconvincing.

In order to give an overall assessment of each DMU from both the optimistic and pessimistic points of view, Entani et al. [2] used model (5) developed by themselves to measure the interval efficiency of each DMU. The results are reported in the fourth column of Table 2, from which can be seen very clearly that their model only successfully identified one DEA efficient DMU, i.e. DMU_A, which has the smallest lower bound efficiency, but failed to identify the other two DEA efficient DMUs. So, the efficient production frontier cannot be determined by their approach.

Since there are four DMUs, i.e. DMU_A, DMU_B, DMU_F, and DMU_J that are identified to be DEA inefficient, in order to determine the lower bound efficiencies of DMUs, four LP models need to be solved for each DMU. Take DMU_A for example. In order to calculate its lower bound efficiency, the following four LP models need to be solved:

$$(\mathrm{LP1}):\quad \theta_{AA}^{L*} = \mathrm{Min}\quad v_1$$
$$\text{s.t.}\quad \begin{cases} u_1 + 8u_2 = 1, \\ v_1 - (u_1 + 8u_2) = 0, \\ u_1, u_2, v_1 \geq 0. \end{cases}$$

$$(\mathrm{LP2}):\quad \theta_{AB}^{L*} = \mathrm{Min}\quad v_1$$
$$\text{s.t.}\quad \begin{cases} u_1 + 8u_2 = 1, \\ v_1 - (2u_1 + 3u_2) = 0, \\ u_1, u_2, v_1 \geq 0. \end{cases}$$

$$(\mathrm{LP3}):\quad \theta_{AF}^{L*} = \mathrm{Min}\quad v_1$$
$$\text{s.t.}\quad \begin{cases} u_1 + 8u_2 = 1, \\ v_1 - (4u_1 + 2u_2) = 0, \\ u_1, u_2, v_1 \geq 0. \end{cases}$$

$$(\mathrm{LP4}):\quad \theta_{AJ}^{L*} = \mathrm{Min}\quad v_1$$
$$\text{s.t.}\quad \begin{cases} u_1 + 8u_2 = 1, \\ v_1 - (7u_1 + u_2) = 0, \\ u_1, u_2, v_1 \geq 0. \end{cases}$$

Each of the above four LP models keeps only one of four DEA inefficient DMUs continuing to be DEA inefficient. The solutions to the above four LP models are as follows:

$$\theta_{AA}^{L*} = 1,\ u_1^* = 1,\ u_2^* = 0\ ,\ v_1^* = 1,$$
$$\theta_{AB}^{L*} = 3/8,\ u_1^* = 0,\ u_2^* = 1/8,\ v_1^* = 3/8,$$
$$\theta_{AF}^{L*} = 1/4,\ u_1^* = 0,\ u_2^* = 1/8,\ v_1^* = 1/4,$$
$$\theta_{AJ}^{L*} = 1/8,\ u_1^* = 0,\ u_2^* = 1/8,\ v_1^* = 1/8.$$

So, the final lower bound efficiency of DMU_A is determined by

$$\theta_A^{L*} = \min\{1, 3/8, 1/4, 1/8\} = 0.1250$$

From the above four sets of input and output weights, it can be seen that only one output (either output 1 or output 2) is effectively used in the computation of lower bound efficiency. Special attention has been paid to the second set of factor weights, i.e. $u_1^* = 0, u_2^* = 1/8, v_1^* = 3/8$, from which we have the following efficiencies for DMU_H, DMU_H, DMU_I, and DMU_J:

$$\theta_F = \theta_H = \theta_I = 3/2, \theta_J = 3$$

They are all greater than one. Such results obviously contradict the assumption that $\max_j\{\sum_{i=1}^m v_i x_{ij} / \sum_{r=1}^s u_r y_{rj}\} = 1$. So, Entani et al.'s [2] solution approach is in fact defective.

As a contrast, we now utilize the bounded DEA model (14) developed in this paper to reevaluate the problem. To do so, we first define the IDMU, which is shown in the last row of Table 1. Its worst relative efficiency is found to be $\varphi^*_{IDMU} = 0.3478$ by running model (12). Running model (14) for each DMU, we get the interval efficiencies of ten DMUs, which are presented in the last column of Table 2, from which it can be seen very clearly that bounded DEA model not only identify the four DEA inefficient DMUs correctly, but also identify the three DEA efficient DMUs fully. The identified DEA efficient DMUs are DMU_A, DMU_E and DMU_J. DMU_A, DMU_B, DMU_F and DMU_J are the four identified DEA inefficient DMUs. Such assessment results are fully consistent with the results obtained by using the traditional CCR model (2) and the worst relative efficiency model (4).

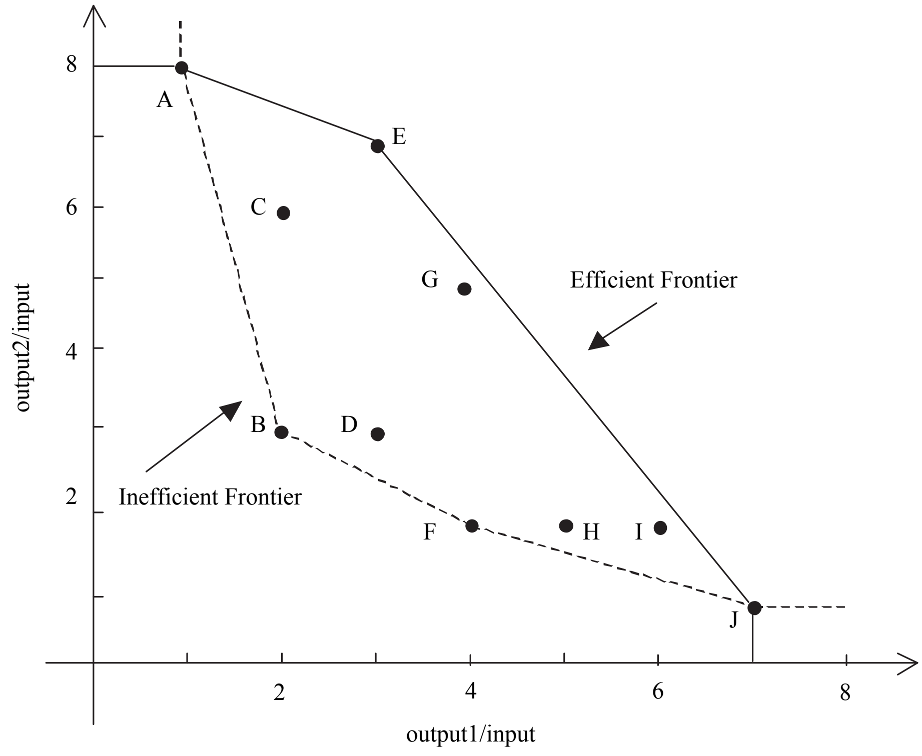

Fig. 1 Efficient and inefficient frontiers for the example

Although DMU_A, DMU_B, DMU_F and DMU_J are all evaluated to be DEA inefficient, due to the differences in their lower bound efficiencies, their performances are in fact not the same. Through comparing their lower bound efficiencies, we find that $DMU_B \succ DMU_F \succ DMU_A \sim DMU_J$. As such, DMU_A, DMU_E and DMU_J are all rated to be DEA efficient, due to the differences in their upper bound efficiencies, their performances are not the same either. Through comparing their upper bound efficiencies, we may arrive at the conclusion that $DMU_A \sim DMU_J \succ DMU_E$. The remaining five DMUs belong to DEA unspecified units. They are all enveloped by the efficient and inefficient production frontiers.

In this example, both DMU_A and DMU_J are evaluated to be DEA inefficient and DEA efficient. This phenomenon shows that the two different production frontiers simultaneously pass through these two specific DMUs (see Fig.1). Usually, DEA efficient units perform well, but this does not mean each DEA efficient unit to be the best. As such, DEA inefficient units usually perform poor, but not every DEA inefficient unit performs the worst. So, when a DMU is both DEA efficient and DEA inefficient, it is likely to mean that the DMU is neither the best nor the worst.

5 Conclusions

Performances of DMUs can be evaluated from different perspectives. Accordingly, the results of such evaluations are often confusing and even contradictory. It is therefore an undeniable necessity to integrate different measures in order to obtain an overall assessment of the performance of each DMU. In this paper, we presented bounded DEA models for measurement of the overall performance of DMUs. It was shown that bounded DEA models have significant advantages over current methods for evaluation of DMUs.

Compared with Entani et al.'s [2] model, the bounded DEA model developed in this paper has some attractive advantages. First of all, it can identify DEA efficient and inefficient DMUs correctly and fully. DEA efficient DMUs form an efficiency frontier, while DEA inefficient DMUs define an inefficiency frontier. All the DEA unspecified DMUs are enveloped by both frontiers. Next, bounded DEA models can make the most of all input and output data in the process of calculating both the upper and lower bound efficiencies of each DMU. So, both the upper and lower bound efficiencies are reasonably determined. Last but not least, the bounded DEA model only needs to solve $(2n+1)$ LP problems. One is solved to determine the worst relative efficiency of the IDMU. The other $2n$ LP problems are solved to compute the upper and lower bounds efficiencies of n DMUs, respectively. The computational burden is substantially reduced.

References

1. Charnes, A., Cooper, W. W., Rhodes, E., (1978). Measuring the efficiency of decision making units. European Journal of Operational Research, 2, 429–444.
2. Entani, T., Maeda, Y., Tanaka, H., (2002). Dual models of interval DEA and its extension to interval data. European Journal of Operational Research, 136, 32–45.
3. Wang, Y. M., Luo, Y., (2006). DEA efficiency assessment using ideal and anti-ideal decision making units. Applied Mathematics and Computation, 173, 902–915.
4. Wang, Y. M., Yang, J. B., (2007). Measuring the performances of decision-making units using interval efficiencies. Journal of Computational and Applied Mathematics, 198, 253–267.
5. Charnes, A., Cooper, W. W., (1962). Programming with fractional functionals. Naval Research Logistics Quarterly, 9, 181–186.
6. Doyle, J. R., Green, R. H., (1994). Efficiency and cross-efficiency in DEA: derivatives meanings and uses. Journal of the Operational Research Society, 45, 567–578.
7. Parkan, C., Wang, Y. M., (2000). Worst Efficiency Analysis Based on Inefficient Production Frontier. Working Paper, Department of Management Sciences, City University of Hong Kong.
8. Azizi, H., Fathi Ajirlu, S., (2010). Measurement of overall performances of decision-making units using ideal and anti-ideal decision-making units. Computers & Industrial Engineering, 59, 411–418.