International Journal of Applied Operational Research
Vol. 6, No. 2, pp. 33-45, Spring 2016

Journal homepage: ijorlu.liau.ac.ir

A Two-stage DEA Model Considering Shared Inputs, Free Intermediate Measures and Undesirable Outputs

S. Fathalikhani*

Received: 2 August 2015 ; **Accepted:** 9 January 2016

Abstract Data envelopment analysis (DEA) has been proved to be an excellent approach for measuring the performance of decision-making units (DMUs) that use multiple inputs to generate multiple outputs. But the allocation problem of shared inputs and undesirable outputs does not arouse attention in this movement. This paper proposes a two-stage DEA model considering simultaneously the structure of shared inputs, additional input in the second stage and part of intermediate products as the final output. In addition, a part of second stage outputs is undesirable which can be fed back as raw materials to the first stage. Cooperative and non-cooperative game theories are discussed in order to determine the upper and lower bounds of the efficiencies of sub-DMUs in different stages to assess the relative performance of the operational units.

Keywords: Shared Inputs, Two-Stage, DEA, Game Theory, Efficiency, Undesirable Outputs.

1 Introduction

Data envelopment analysis (DEA) was introduced by Charnes, Cooper, and Rhodes in 1978. It is a non-parametric linear programming based technique for evaluating the relative efficiency of a set of decision-making units (DMUs). Since the work on CCR model of Charnes et al. [1], large number of research on DEA models has been developed, such as BCC model [2], FDH model [3], SBM model [4], EBM model[5], RBM model [6]and NEBM [7]. As indicated in [8], DEA can be applied to identify sources of inefficiency, rank the DMUs, evaluate management, evaluate the effectiveness of program or policies, create a quantitative basis for reallocating resources, etc. Over the last decade, DEA has gained considerable attention as a managerial tool for measuring the performance of DMUs.

In conventional DEA, DMUs are treated as a black-box in the sense that internal structures are generally ignored, and the performance of a DMU is assumed to be a function of the chosen inputs and outputs. So, these DEA models may show a black-box unit as an efficient, while it contains some inefficient sub-processes. Otherwise, more and more researchers (see example [9]; [10]; [11]; etc.) attempt to get into the inside of the ‘‘black box’’ by paying attention to the internal structure of the DMUs. The models developed in this approach are so-called network DEA models which consider the process within a DMU as composed by several sub-processes or stages, every stage characterized by its own inputs and outputs, and related by intermediate flows [12].

* **Corresponding Author.** (✉)
E-mail: somayefathalikhani@gmail.com (S. Fathalikhani)

S. Fathalikhani
Ph.D candidate, Department of Industrial Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran.

Recently, a number of studies have looked at DMUs that have a two-stage network structure where in addition to the inputs and outputs, a set of intermediate measures exists in-between the two stages. These intermediate measures are the outputs from the first stage that become the only inputs to the second stage. By modeling the relations between serial stages, two-stage DEA models are able to evaluate the overall efficiency of the DMUs and decompose it into the efficiency of each stage. In consequence, the two-stage DEA models are capable of providing more specific information about the efficiency or inefficiency of internal operations within the DMUs.

Several studies have been reported to deal with two-stage DEA models and its extensions to more general cases from different points of view. Seiford and Zhu [13] deal with two-stage systems to calculate the efficiency score of commercial banks of US. Zhu [14] evaluated the efficiency scores of the best 500 companies by using the same two-stage structure. Fare [15]introduced a method to analyze the performance of each sub-processes by considering intermediate products. Kao and Hwang [16] proposed the standard DEA models by considering the series relation between the stages of network systems. Kao [17]introduced a relational method for evaluating general network systems, and then, by introducing dummy processes transform the systems into series processes in which each process comprises of parallel processes. Kao and Hwang [18] presented a model to indicate relevance between the efficiency of the system and its processes. Zhu et al. [19] showed that the multiplier and envelopment network DEA models have different results in presenting divisional efficiency. Additionally, they mentioned that proper benchmarks cannot be derived from most of the network DEA models. Kao [20]considered general multi-stage systems as the systems in which exogenous inputs are consumed in addition to intermediate products. Kao [21]proposed a general SBM model for evaluating the efficiency score of network systems in which the system efficiency is decomposed into a weighted average of processes efficiency. Kao [22]reviews some studies on network DEA. Jianfeng [23] proposes a two-stage DEA model considering simultaneously the structure of shared inputs and intermediate measures in efficiency evaluation and decomposition.

The above-mentioned studies on network DEA are very significant, but they do not consider shared inputs and undesirable outputs which characterize the relations between the two stages and influence the overall efficiency decomposition. The paper proposes a two-stage DEA model in which the intermediate measures from the first stage fall into the inputs to the second stage and the final outputs for the market, and the proportion of the division is freely determined by decision makers. At the same time, the proposed model takes into consideration the structure of inputs by differentiating between the inputs devoted to each stage and the inputs shared by two stages. Parts of outputs from the second stage are wastages that can be fed back as inputs to the first stage.

This paper is structured as followed. Section 2 develops a non-cooperative and cooperative model to measure the efficiency of the proposed two-stage model. A numerical example is illustrated to justify the new model in section 3. Conclusions and directions for future research are provided in the last section.

2 The Models

Suppose that there are a set of n DMUs denoted by $DMU_j\ (j=1,\dots,n)$ which is illustrated in Fig. 1. Each $DMU_j\ (j=1,\dots,n)$ has $m$initial inputs denoted by$x_{ij}, (i = 1, \dots, m)$to the whole process and H additive inputs denoted by $x_{hj}, (h=1,\ldots,H)$. Parts of these m inputs are the only inputs to the first stage while other inputs are used or shared as inputs in both stages. We denote these two types of inputs as $x_{i_1 j}\,(i_1 \in I_1)$ and shared inputs $x_{i_2 j}\,(i_2 \in I_2)$, respectively, where $I_1 \cup I_2 = \{1,2,\ldots,m\}$ and $I_1 \cap I_2 = \varnothing$.

Since inputs $i_2 \in I_2$ are shared by both stages, we assume that all $x_{i_2 j}\,(i_2 \in I_2)$ are divided into $\alpha_{i_2 j} x_{i_2 j}$ and $(1-\alpha_{i_2 j})x_{i_2 j}\,(0 \le \alpha_{i_2 j} \le 1)$, corresponding to the portions of shared inputs used by the first and second stage, respectively. Similar to the constraints in [24], all $\alpha_{i_2 j}\,(i_2 \epsilon I_2,\, j=1,\ldots,n)$ will be required to be within certain intervals, namely $L^1_{i_2 j} \le \alpha_{i_2 j} \le L^2_{i_2 j}$.

Assume that each $DMU_j\,(j=1,\ldots,n)$ has D outputs denoted by $z_{dj}\,(d=1,\ldots,D)$ from the first stage, and s final outputs denoted by $y_{rj}\,(r=1,\ldots,s)$ and G outputs denoted by $f_{gj}\,(g=1,\ldots,G)$ from the second stage. Part of intermediate products by the sub-DMU in stage 1 is consumed by the sub-DMU in stage 2, and the rest of them can turn out to be final output in the market. The portions of intermediate measures is denoted by $\beta_{dj} z_{dj}$ and the portions of exited outputs by $(1-\beta_{dj})z_{dj}$, where $0 < \beta_{dj} \le 1$ and $H^1_{dj} \le \beta_{dj} \le H^2_{dj}$. It should be noted that $f_{gj}\,(g=1,\ldots,G)$, outputs from the second stage, are wastages that can be fed back as inputs to the first stage.

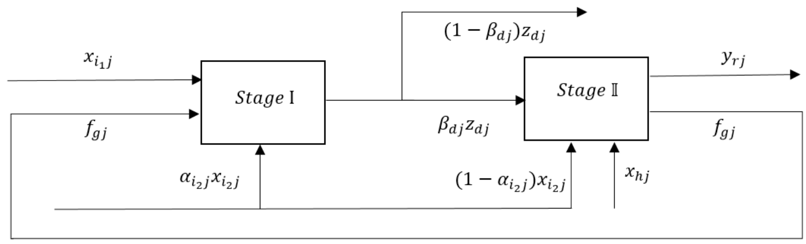

Fig. 1 Two-stage network process

2.1 The Non-Cooperative Model

In this section, according to the concepts of the leader-follower or the Stackelberg game theory[25], we will discuss the efficiencies of the sub-DMUs under the non-cooperative condition, and obtain the upper and lower bounds of their efficiencies.

2.1.1 First Stage Dominates the System, While the Second Follows

The efficiency of the first stage is evaluated as follows:

$$U_1 = \max \frac{\sum_{d=1}^{D} \eta_d z_{dp}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp}}$$

$$s.t.$$

$$\frac{\sum_{d=1}^{D} \eta_d z_{dj}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 j} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 j} x_{i_2 j} + \sum_{g=1}^{G} w_g f_{gj}} \le 1, \quad j = 1, \ldots, n \tag{1}$$

$$L_{i_2 j}^{1} \le \alpha_{i_2 j} \le L_{i_2 j}^{2}, i_2 \in I_2, j = 1, \ldots, n,$$

$$\eta_d, w_g, v_{i_1}, v_{i_2} \ge \varepsilon, d = 1, \ldots, D, g = 1, \ldots, G, i_1 \epsilon I_1, i_2 \epsilon I_2.$$

Model (1) can be transformed into the following linear Model, by using the [26] transformation. By model (2) the upper efficiency of first stage can be achieved.

$$\theta_1^U = \max \sum_{d=1}^{D} \eta_d z_{dp}$$

$$s.t.$$

$$\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} = 1,$$

$$\sum_{d=1}^{D} \eta_d z_{dj} - \sum_{i \epsilon I_1} v_{i_1} x_{i_1 j} - \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 j} x_{i_2 j} - \sum_{g=1}^{G} w_g f_{gj} \le 0, \tag{2}$$

$$j = 1, \ldots, n$$

$$L_{i_2 j}^{1} \le \alpha_{i_2 j} \le L_{i_2 j}^{2}, i_2 \in I_2, j = 1, \ldots, n,$$

$$\eta_d, w_g, v_{i_1}, v_{i_2} \ge \varepsilon, d = 1, \ldots, D, g = 1, \ldots, G,$$

$$i_1 \epsilon I_1, i_2 \epsilon I_2.$$

When the first stage is assumed the leader, the efficiency of the second stage (follower) is computed, subject to the requirement that the leader's efficiency stays fixed. The following model calculates the corresponding efficiency of second stage.

$$\theta_2^L = \max \frac{\sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp}}{\sum_{i \int I_2} v_{i_2} (1 - \alpha_{i_2 p}) x_{i_2 p} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp}}$$

$$s.t.$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj}}{\sum_{i \int I_2} v_{i_2} (1 - \alpha_{i_2 p}) x_{i_2 j} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dj} + \sum_{g=1}^{G} q_h x_{hj}} \le 1, \tag{3}$$

$$j = 1, \ldots, n$$

$$\sum_{d=1}^{D} \eta_d z_{dp} = \theta_1^{U*},$$

$$\begin{aligned}
&\sum_{i \in I_1} v_{i_1} x_{i_1 p} + \sum_{i \in I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} = 1, \\
&\sum_{d=1}^{D} \eta_d z_{dj} - \sum_{i \in I_1} v_{i_1} x_{i_1 j} - \sum_{i \in I_2} v_{i_2} \alpha_{i_2 j} x_{i_2 j} - \sum_{g=1}^{G} w_g f_{gj} \leq 0 \ , \\
&j = 1, \ldots, n \\
&L_{i_2 j}^{1} \leq \alpha_{i_2 j} \leq L_{i_2 j}^{2}, i_2 \in I_2, j = 1, \ldots, n, \\
&H_{dj}^{1} \leq \beta_{dj} \leq H_{dj}^{2}, d = 1, \ldots, D, j = 1, \ldots, n, \\
&u_r, \eta_d, w_g, q_h, v_{i_1}, v_{i_2} \geq \varepsilon \ , r = 1, \ldots, s \ , d = 1, \ldots, D, \\
&g = 1, \ldots, G, h = 1, \ldots, H, i_1 \in I_1, i_2 \in I_2
\end{aligned}$$

Via the Charnes-Cooper transformation, model (3) is transformed as:

$$\theta_2^L = max \sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp}$$

$s.t.$

$$\begin{aligned}
&\sum_{i \epsilon I_2} v_{i_2} (1 - \alpha_{i_2 p}) x_{i_2 p} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp} = \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj} - \sum_{i \epsilon I_2} v_{i_2} (1 - \alpha_{i_2 p}) x_{i_2 j} - \\
&\sum_{d=1}^{D} \eta_d \beta_{dp} z_{dj} - \sum_{g=1}^{G} q_h x_{hj} \leq 0, \quad j = 1, \ldots, n, \\
&\sum_{d=1}^{D} \eta_d z_{dp} = \theta_1^{U*}, \\
&\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} = 1, \\
&\sum_{d=1}^{D} \eta_d z_{dj} - \sum_{i \epsilon I_1} v_{i_1} x_{i_1 j} - \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 j} x_{i_2 j} - \sum_{g=1}^{G} w_g f_{gj} \\
&j = 1, \ldots, n \\
&L_{i_2 j}^{1} \leq \alpha_{i_2 j} \leq L_{i_2 j}^{2}, i_2 \in I_2, j = 1, \ldots, n, \\
&H_{dj}^{1} \leq \beta_{dj} \leq H_{dj}^{2}, d = 1, \ldots, D, j = 1, \ldots, n, \\
&u_r, \eta_d, w_g, q_h, v_{i_1}, v_{i_2} \geq \varepsilon \ , r = 1, \ldots, s \ , d = 1, \ldots \\
&g = 1, \ldots, G, h = 1, \ldots, H, i_1 \epsilon I_1, i_2 \epsilon I_2
\end{aligned} \tag{4}$$

Where θ_2^L is the lower efficiency of the second stage.

2.1.2 Second Stage Dominates the System, While the First Follows

With the similar manner in 2.1.1, we assume the second stage to be the leader and calculate the regular DEA efficiency for stage2, using the appropriate CCR model

$$\theta_2^U = \max \frac{\sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp}}{\sum_{i \in I_2} v_{i_2}(1-\alpha_{i_2p}) x_{i_2p} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp}}$$

$$\begin{aligned}
s.t.\quad & \frac{\sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj}}{\sum_{i \in I_2} v_{i_2}(1-\alpha_{i_2p}) x_{i_2j} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dj} + \sum_{g=1}^{G} q_h x_{hj}} \le 1, \\
& j = 1,\ldots,n \\
& L_{i_2j}^1 \le \alpha_{i_2j} \le L_{i_2j}^2, i_2 \in I_2, j = 1,\ldots,n, \\
& H_{dj}^1 \le \beta_{dj} \le H_{dj}^2, d = 1,\ldots,D, j = 1,\ldots,n, \\
& u_r, \eta_d, w_g, q_h, v_{i_2} \ge \varepsilon, r = 1,\ldots,s, d = 1,\ldots,D, \\
& g = 1,\ldots,G, h = 1,\ldots,H, i_2 \in I_2.
\end{aligned} \tag{5}$$

Model (5) now can be transformed via the Charnes-Cooper transformation as follows:

$$\theta_2^U = \max \sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp}$$

$$\begin{aligned}
s.t.\quad & \sum_{i \in I_2} v_{i_2}(1-\alpha_{i_2p}) x_{i_2p} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj} - \sum_{i \in I_2} v_{i_2}(1-\alpha_{i_2p}) x_{i_2j} - \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dj} \\
& - \sum_{g=1}^{G} q_h x_{hj} \le 0 \quad j = 1,\ldots,n, \\
& L_{i_2j}^1 \le \alpha_{i_2j} \le L_{i_2j}^2, i_2 \in I_2, j = 1,\ldots,n, \\
& H_{dj}^1 \le \beta_{dj} \le H_{dj}^2, d = 1,\ldots,D, j = 1,\ldots,n, \\
& u_r, \eta_d, w_g, q_h, v_{i_2} \ge \varepsilon, r = 1,\ldots,s, d = 1,\ldots,D, \\
& g = 1,\ldots,G, h = 1,\ldots,H, i_2 \in I_2.
\end{aligned} \tag{6}$$

According to the above linear programming model, the optimum efficiency of the second stage is obtained.

The lower efficiency of the first stage as the follower one can be calculated as follows:

$$
\begin{aligned}
&\theta_1^L = \max \frac{\sum_{d=1}^{D}\eta_d z_{dp}}{\sum_{i \in I_1} v_{i_1} x_{i_1p} + \sum_{i \in I_2} v_{i_2}\alpha_{i_2p} x_{i_2p} + \sum_{g=1}^{G} w_g f_{gp}} \\
&s.t. \\
&\qquad \frac{\sum_{d=1}^{D}\eta_d z_{dj}}{\sum_{i \in I_1} v_{i_1} x_{i_1j} + \sum_{i \in I_2} v_{i_2}\alpha_{i_2j} x_{i_2j} + \sum_{g=1}^{G} w_g f_{gj}} \le 1, \\
&\qquad j = 1,\dots,n \\
&\qquad \sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp} = \theta_2^{U*}, \\
&\qquad \sum_{i \in I_2} v_{i_2}(1-\alpha_{i_2p}) x_{i_2p} + \sum_{d=1}^{D}\eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp} = 1, \qquad\qquad (7) \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj} - \sum_{i \in I_2} v_{i_2}(1-\alpha_{i_2p}) x_{i_2j} - \\
&\qquad \sum_{d=1}^{D}\eta_d \beta_{dp} z_{dj} - \sum_{g=1}^{G} q_h x_{hj} \le 0 \quad j = 1,\dots,n, \\
&\qquad L_{i_2j}^1 \le \alpha_{i_2j} \le L_{i_2j}^2, i_2 \in I_2, j = 1,\dots,n, \\
&\qquad H_{dj}^1 \le \beta_{dj} \le H_{dj}^2, d = 1,\dots,D, j = 1,\dots,n, \\
&\qquad u_r, \eta_d, w_g, q_h, v_{i_1}, v_{i_2} \ge \varepsilon, r = 1,\dots,s, d = 1,\dots,D, \\
&\qquad g = 1,\dots,G, h = 1,\dots,H, i_1 \in I_1, i_2 \in I_2.
\end{aligned}
$$

The lower efficiency of the first stage is obtained by model 7 with the restriction that the second stage score have already been determined and cannot be decreased from that value, Where θ_2^{U*} is the optimum efficiency of the second stage. By using the same transformation techniques, the model (7) is converted to

$$
\begin{aligned}
&\theta_1^L = \max \sum_{d=1}^{D}\eta_d z_{dp} \\
&s.t. \\
&\qquad \sum_{i \in I_1} v_{i_1} x_{i_1p} + \sum_{i \in I_2} v_{i_2}\alpha_{i_2p} x_{i_2p} + \sum_{g=1}^{G} w_g f_{gp} = 1 \\
&\qquad \sum_{d=1}^{D}\eta_d z_{dj} - \sum_{i \in I_1} v_{i_1} x_{i_1j} - \sum_{i \in I_2} v_{i_2}\alpha_{i_2j} x_{i_2j} - \sum_{g=1}^{G} w_g f_{gj} \le 0 \qquad\qquad (8) \\
&\qquad j = 1,\dots,n \\
&\qquad \sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp} = \theta_2^{U*}, \\
&\qquad \sum_{i \in I_2} v_{i_2}(1-\alpha_{i_2p}) x_{i_2p} + \sum_{d=1}^{D}\eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp} = 1,
\end{aligned}
$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj} - \sum_{i \epsilon I_2} v_{i_2} (1-\alpha_{i_2 p}) x_{i_2 j} - \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dj},$$
$$-\sum_{g=1}^{G} q_h x_{hj} \le 0 \quad j = 1,\ldots,n,$$
$$L^1_{i_2 j} \le \alpha_{i_2 j} \le L^2_{i_2 j}, i_2 \in I_2, j = 1,\ldots,n,$$
$$H^1_{dj} \le \beta_{dj} \le H^2_{dj}, d = 1,\ldots,D, j = 1,\ldots,n,$$
$$u_r, \eta_d, w_g, q_h, v_{i_1}, v_{i_2} \ge \varepsilon, r = 1,\ldots,s, d = 1,\ldots,D,$$
$$g = 1,\ldots,G, h = 1,\ldots,H, i_1 \epsilon I_1, i_2 \epsilon I_2.$$

2.2 The Cooperative Model

The concept of cooperative game theory is showed by (Liang, et al, 2008), the two stage process can be viewed as one where the stages jointly determine a set of optimal weights on the intermediate factors to maximize their efficiency scores.

It is assumed that the worth or value accorded to the intermediate variable is the same regardless of whether they are viewed as inputs or outputs (Liang, et al, 2008). The cooperative efficiency model of two-stage production process illustrated in fig.1 can be described as

$$\theta_t = w_1 \theta_1 + w_2 \theta_2$$
$$\theta_1 = \max \frac{\sum_{d=1}^{D} \eta_d z_{dp}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp}}$$
$$\theta_2 = \max \frac{\sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp}}{\sum_{i \epsilon I_2} v_{i_2} (1-\alpha_{i_2 p}) x_{i_2 p} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp}}$$
$$w_1 = \frac{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp}}$$
$$w_2 = \frac{\sum_{i \epsilon I_2} v_{i_2} (1-\alpha_{i_2 p}) x_{i_2 p} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp}}$$
$$\theta_t = \frac{\sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d z_{dp}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h x_{hp}}$$

Where θ_1 and θ_2 are the ratio efficiencies for stages 1 and 2, respectively. Therefore, the cooperative efficiency model of two-stage process is formulated as follows:

$$\theta_t^* = max \frac{\sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d z_{dp}}{\sum_{i\epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i\epsilon I_2} v_{i_2} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_{i}}$$

$$\begin{aligned}
& s.t. \\
& \frac{\sum_{d=1}^{D} \eta_d z_{dj}}{\sum_{i\epsilon I_1} v_{i_1} x_{i_1 j} + \sum_{i\epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 j} + \sum_{g=1}^{G} w_g f_{gj}} \le 1, \quad j = 1,\ldots n, \\
& \frac{\sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj}}{\sum_{i\epsilon I_2} v_{i_2} (1-\alpha_{i_2 p}) x_{i_2 j} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dj} + \sum_{g=1}^{G} q_h x_{hj}} \le 1, \quad j = 1,\ldots n, \\
& H_{dj}^1 \le \beta_{dj} \le H_{dj}^2, d = 1,\ldots,D, j = 1,\ldots,n, \\
& u_r, \eta_d, w_g, q_h, v_{i_1}, v_{i_2} \ge \varepsilon, r = 1,\ldots,s, d = 1,\ldots,D, \\
& g = 1,\ldots,G, h = 1,\ldots,H, i_1 \epsilon I_1, i_2 \epsilon I_2.
\end{aligned} \tag{9}$$

By applying the charnes- cooper transformation, model (9) can be transformed into

$$\begin{aligned}
& \theta_t^* = \max \sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d z_{dp} \\
& s.t. \\
& \sum_{i\epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i\epsilon I_2} v_{i_2} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} \\
& + \sum_{g=1}^{G} q_h x_{hp} = 1, \\
& \sum_{d=1}^{D} \eta_d z_{dj} - \sum_{i\epsilon I_1} v_{i_1} x_{i_1 j} - \sum_{i\epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 j} - \sum_{g=1}^{G} w_g f_{gj} \le 0, \\
& j = 1,\ldots,n \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{G} w_g f_{gj} - \sum_{i\epsilon I_2} v_{i_2} (1-\alpha_{i_2 p}) x_{i_2 j} - \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dj} \\
& - \sum_{g=1}^{G} q_h x_{hj} \le 0, \quad j = 1,\ldots,n, \\
& H_{dj}^1 \le \beta_{dj} \le H_{dj}^2, d = 1,\ldots,D, j = 1,\ldots,n, \\
& u_r, \eta_d, w_g, q_h, v_{i_1}, v_{i_2} \ge \varepsilon, r = 1,\ldots,s, d = 1,\ldots,D, \\
& g = 1,\ldots,G, h = 1,\ldots,H, i_1 \epsilon I_1, i_2 \epsilon I_2,
\end{aligned} \tag{10}$$

Now, the cooperative efficiency model of first stage denoted by θ_1 is formulated as follows;

$$
\begin{aligned}
&\theta_1^* = \max \ \frac{\sum_{d=1}^{D} \eta_d z_{dp}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp}} \\
&s.t. \\
&\qquad \frac{\sum_{d=1}^{D} \eta_d z_{dj}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 j} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 j} x_{i_2 j} + \sum_{g=1}^{G} w_g f_{gj}} \le 1 \quad j = 1, \dots, n, \\
&\qquad \frac{\sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d z_{dp}}{\sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d \beta_{dp} z_{dp} + \sum_{g=1}^{G} q_h \lambda} \\
&\qquad L_{i_2 j}^1 \le \alpha_{i_2 j} \le L_{i_2 j}^2, i_2 \in I_2, j = 1, \dots, n, \\
&\qquad u_r, \eta_d, w_g, q_h, v_{i_1}, v_{i_2} \ge \varepsilon, r = 1, \dots, s, d = 1, \dots, D, \\
&\qquad g = 1, \dots, G, h = 1, \dots, H, i_1 \epsilon I_1, i_2 \epsilon I_2.
\end{aligned}
\tag{11}
$$

Where $\theta_t^*$is overall efficiency of two stage process.By using same transformation, model (11) can be transformed into:

$$
\begin{aligned}
&\theta_1^* = \max \sum_{d=1}^{D} \eta_d z_{dp} \\
&s.t. \\
&\qquad \sum_{i \epsilon I_1} v_{i_1} x_{i_1 p} + \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 p} x_{i_2 p} + \sum_{g=1}^{G} w_g f_{gp} = 1, \\
&\qquad \sum_{d=1}^{D} \eta_d z_{dj} - \sum_{i \epsilon I_1} v_{i_1} x_{i_1 j} - \sum_{i \epsilon I_2} v_{i_2} \alpha_{i_2 j} x_{i_2 j} - \sum_{g=1}^{G} w_g f_{gj} \le 0, \\
&\qquad j = 1, \dots, n, \\
&\qquad \sum_{r=1}^{s} u_r y_{rp} - \sum_{g=1}^{G} w_g f_{gp} + \sum_{d=1}^{D} \eta_d z_{dp} - \theta_t^* = 0, \\
&\qquad L_{i_2 j}^1 \le \alpha_{i_2 j} \le L_{i_2 j}^2, i_2 \in I_2, j = 1, \dots, n, \\
&\qquad u_r, \eta_d, w_g, v_{i_1}, v_{i_2} \ge \varepsilon, r = 1, \dots, s, d = 1, \dots, D, \\
&\qquad g = 1, \dots, G, i_1 \epsilon I_1, i_2 \epsilon I_2.
\end{aligned}
\tag{12}
$$

As the overall efficiency of the $DMU_p$is the weighted arithmetic mean of the efficiencies of the two stages, the efficiency for the second stage can be calculated as$\theta_2^* = \frac{\theta_t^* - w_1^* \theta_1^*}{w_2^*}$, where w_1^* and w_2^* represent the optimal weights obtained from the model (10).

3 An Illustrative Application

After formulating the proposed model a numerical example is employed to explain it. Suppose there is a two-stage produce process in which there are three types of inputs; raw material to first stage to produce product A (x_1), raw material to second stage to produce product B (x_3), and labor shared by two stages (x_2). The output from the first stage is number of product A (z). Some part of the intermediate product A shipped as the final output (e.g. those parts are marketed). The other parts of intermediate products A are processed further in the second stage. The second stage has two outputs sales (y) and wastages (f) of production process that can be fed back to the first stage as raw material. Table 1 provides the data set contained 10 DMU ($DMU_j, j = 1,\ldots,10$).

Table 1 data set

DMU	Raw material x_1	Labor x_2	Product A z	Raw material x_3	Profit y	Wastage f	α_2	β
DMU_1	242	118	168	170	153	48	0.76	0.98
DMU_2	247	123	106	184	251	40	0.58	0.99
DMU_3	195	179	93	139	142	19	0.45	0.95
DMU_4	305	215	232	198	397	24	0.32	0.88
DMU_5	280	144	272	125	125	57	0.29	0.96
DMU_6	144	105	251	207	108	32	0.35	0.88
DMU_7	289	98	162	234	299	55	0.54	0.95
DMU_8	185	163	198	120	250	45	0.26	0.73
DMU_9	389	156	265	117	215	38	0.16	0.92
DMU_{10}	179	132	189	103	116	18	0.42	0.92

Table 2 presents the cooperative efficiencies and the relative efficiencies of the two stages. For calculation, $\varepsilon = 0.001$ is chosen. The DEA models are coded using LINGO 11 software. The first three columns of the table 2represent the total optimal efficiency of model (10) along with the stages' optimal efficiencies. The rank of each DMU is indicated in parentheses. As can be seen in Table 2, because there do not exist any DMUs with two efficient stages, therefore there are not any efficient DMUs.

The last two columns show the optimal proportion of each stage in total optimal efficiency. These indicate that the second stage is more important (the second stage is treated as the leader). For example, DMU_6 and DMU_{10} are efficient in first stage, but because of low efficiency in second stage, corresponding performance rating become five and six, respectively.

Table 2 the result based on cooperative model

DMU	θ^*_{Total}	$\theta^*_{\text{Stage I}}$	$\theta^*_{\text{Stage II}}$	w_1^*	w_2^*
DMU_1	0.4666 (8)	0.4192	0.5131	0.4950	0.5050
DMU_2	0.6535 (4)	0.3055	0.9477	0.4581	0.5419
DMU_3	0.4481 (9)	0.4531	0.4458	0.3173	0.6827
DMU_4	0.8031 (2)	0.8706	0.7504	0.4386	0.5614
DMU_5	0.4305 (10)	0.7048	0.2195	0.4348	0.5652
DMU_6	0.5389 (5)	1	0.2496	0.3855	0.6145

DMU	θ^*_{Total}	$\theta^*_{\text{Stage I}}$	$\theta^*_{\text{Stage II}}$	w^*_1	w^*_2
DMU_7	0.6883 (3)	0.4076	0.9198	0.4520	0.5480
DMU_8	0.8182 (1)	0.6598	0.8933	0.3215	0.6785
DMU_9	0.5174 (7)	0.7824	0.2570	0.4956	0.5044
DMU_{10}	0.5344 (6)	1	0.3587	0.2740	0.7260

4 Conclusions

The current paper tries to enrich the previous two-stage DEA modeling and applications literature by providing a model with shared inputs, free intermediate measures, and undesirable final outputs. The two-stage network analyzed structure distinguishes between the intermediate measures which become inputs to the second stage and that turn out to be final output in the market, It also considers all kinds of inputs to evaluate the system efficiency; initial inputs to the first stage, shared inputs between the two stages, additive inputs to the second stage. Part of outputs from the second stage, are wastages that can be fed back as inputs to the first stage. In reality, many organizations actually have this kind of structure.

The aim of this paper is to provide an analytical game-theory framework to calculate maximize the overall efficiency of the DMU and sub-DMUs under cooperative or non-cooperative conditions. By the proposed model, it can be possible to find a set of appropriate proportion for the sharing the inputs between the stages and to decide whether intermediate products should be sold at the split-off point or processed further. A simple numerical example has been used to demonstrate the theoretical contributions of the current paper.

The limitations of the conceptual and analytical frameworks provide potential starting points for future work. The current models are under the assumption of CRS (constant return to scale), how to modify these models for general network structure by VRS (variable return to scale) assumption is also a direction for future research. Another interesting direction of research is that of modeling the proposed structure with a perspective of dynamic effects and investigate the relative efficiency of each stage. Finally, in future empirical analyzes on this subject, the proposed framework can also be applied to other complex production processes or service processes.

References

1. Charnes, A., Cooper, W. W., Rhodes E. (1978). Measuring the efficiency of decision making units. European Journal of Operational Research, 2(6), 429-444.
2. Banker, R. D., Charnes, A., Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management Science, 30(9), 1078-1092.
3. Petersen, N.C. (1990). Data envelopment analysis on a relaxed set of assumptions. Management Science, 36, 305-313.
4. Tone, K. (2001). A slack-based measure of efficiency in data envelopment analysis. European Journal of Operational Research, 130, 498-509.
5. Tone, K., Tsutsui, M. (2010). An epsilon-based measure of efficiency in DEA: A Third pole of technical efficiency. European Journal of operational Research, 207(3), 1554-1563.
6. Washio, S., Yamada, S. (2013). Evaluation method based on ranking in data envelopment analysis. Expert Systems with Applications, 40, 257-262.
7. Tavana, M., Mirzagoltabar, H., Mirhedayatian, S. M., Saen, R. F., Azadi, M. (2013). A new network epsilon-based DEA model for supply chain performance evaluation. Computers & Industrial Engineering, 66, 501-513.
8. Golang, B., Roll, Y. (1989). An application procedure for DEA. Omega the International Journal of Management Science, 17, 237-250.

9. Hahn, J. S., Kim, D. K., Kim, H. C., Lee, C. (2013). Efficiency analysis on bus companies in Seoul City using a network DEA model. Journal of Civil Engineering, 17(16), 1480-1488.
10. Wade, D. C., Zhu, J., Bi, G., Yang, F. (2010). Network DEA: Additive efficiency. European Journal of Operational Research, 207, 1122-1129.
11. Wei, Q. L., Chang, T. S. (2011). Optimal system design series-network DEA models. Computer Software, Data Handling and Applications, 62(6), 1109-1119.
12. Mirhedayatian, S. M., Azadi, M., Saen, R. F. (2014). A novel network data envelopment analysis model for evaluating green supply chain management. International Journal of Production Economics, 147, 544-554.
13. Seiford, L. M., Zhu, J. (1999). Profitability and marketability of the top 55 U.S. commercial Bank. Management Science, 45 1270-1288.
14. Zhu, J. (2000). Multi-factor performance measure model with an application to Fortune 500 companies. European Journal of Operational Research, 123 105-124.
15. Fare, R., Grosskopf, SH., Whittaker, G. (2000). Network DEA. Socio-Economic Planning Sciences, 34, pp. 35-49.
16. Kao, C., Hwang, S. N. (2008). Efficiency decomposition in two-stage data envelopment analysis: an application to non-life insurance companies in Taiwan. European Journal of Operational Research, 85, 418-429.
17. Kao, C. (2009). Efficiency decomposition in network data envelopment analysis: A relational model. European Journal of Operational Research, 192, 949-962.
18. Kao, C., Hwang, S. N. (2010). Efficiency measurement for network systems: IT impact on firm performance. Decision Support Systems, 48, 437-446.
19. Chen, Y., Cook, W. D., Kao, Ch., Zhu, J. (2013). Network DEA pitfalls: Divisional efficiency and frontier projection under general network structures. European Journal of Operational Research, 226, 507-515.
20. Kao, Ch. (2014). Efficiency decomposition for general multi-stage systems in data envelopment analysis. European Journal of Operational Research, 232, 117-224.
21. Kao, Ch. (2014). Efficiency decomposition in network data envelopment analysis with slacks-based measures. omega, 45, 1-6.
22. Kao, Ch. (2014). Network data envelopment analysis: A review. European Journal of Operational Research, 239, 1-16.
23. Jianfeng, M. A. (2015). A two-stage DEA model considering shared inputs and free intermediate measures. Expert Systems with Applications, 42 ,4339-4347.
24. Cook, W. D., Hababou, M. (2001). Sales performance measurement in bank branches. omega , 29 (2), 299-307.
25. Liang, L., Cook, W. D., Zhu, J. (2008). DEA models for two-stage processes: Game approach and efficiency decomposition. Naval Research Logistics, 55, 643-653.
26. Charnes, A., Cooper, W. W., (1962). Programming with linear fractional functionals. Naval Research logistics quarterly, 9, 181-186.
27. Wang, C. H., Gopal, R., Zionts, S., (1997). Use of data envelopment analysis in assessing information technology impact on firm performance. Annals of Operations Research, 73, 191-213.
28. Liang, L., Cook, W. D., Zhu, J., (2008). DEA models for two-stage processes: Game approach and efficiency decomposition. Naval Research Logistics, 55, 643-653.