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On the hybrid conjugate gradient method for solving fuzzy optimization problem

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Abstract In this paper we consider a constrained optimization problem where the objectives are fuzzy functions (fuzzy-valued functions). Fuzzy constrained Optimization (FO) problem plays an important role in many fields, including mathematics, engineering, statistics and so on. In the other side, in the real situations, it is important to know how may obtain its numerical solution of a given interesting problem. In this paper we defined Necessary and sufficient optimality conditions for constrained fuzzy optimization problem and a Hybrid Conjugate Gradient (HCG) algorithm was proposed for solving FOP. The parametric form of the function was translated into an equivalent Fuzzy constrained Optimization (FO) problem. Then, HCG algorithm was applied to solve the corresponding optimization problem.
Keywords: Fuzzy constrained Optimization, Hybrid Conjugate Gradient Algorithm, constrained Optimization Problem, Steepest Descent Algorithm, Generalized Hukuhara differentiability

1. Introduction

Fuzzy optimization problems have been studied by many researchers in several directions with a lot of applications. A hybrid method of the Polak-Ribière-Polyak (PRP) method and the Fletcher & Reeves (FR) method is proposed for constrained optimization problems.
We consider the following nonlinear fuzzy optimization problem

$$\begin{aligned} \min \quad & \tilde{f} = \tilde{f}\left(x_1,\ldots,x_n\right) \\ \text{s.t} \quad & \tilde{g}_j\left(x\right) \leq \tilde{0},\ j=1,\ldots,m \end{aligned}$$

It is well known that there are many methods for solving fuzzy optimization problems, where the conjugate gradient(CG) method is a powerful line search method because of its simplicity and its very low memory requirement, especially for the large scale optimization problem, which can avoid, like steepest descent method, the computation and storage of some matrices associated with the Hessian of objective functions.

Zadeh [3] first introduced the concept of fuzzy numbers and investigated their arithmetic operation. The fuzzy numbers are widely applied in nonlinear function whose parameters are all or partially represented by fuzzy numbers etc. Fuzzy Nonlinear Equation (FNLE) is useful in solving problems which are difficult, impossible to solve due to the imprecise, subjective nature of the problem formulation or have an accurate solution.

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In order to obtain the numerical solutions of the class of nonlinear equation with fuzzy numbers, Abbasbandy etc. in [1,2] presented Newton algorithm and Steepest Descent (SD) algorithm, respectively. Convergence of Newton algorithm in [1] is dependent on initial approximate solution. If the initial solution is close to the exact one of the nonlinear function, Newton algorithm can converge rapidly, or it may be divergent. Convergence of SD algorithm in [2] has no dependence on initial solution, but its convergence rate is slow. It is well known that Conjugate Gradient (CG) algorithm is an effective method for solving FO problem. Its program design is simple and it need less computer memory. HCG algorithm is an important development of CG algorithm, and it has some advantages which do not belong to certain single CG algorithm. Therefore, it is an interesting problem to study how to use HCG algorithm to solve the following FNLE:

$$F(x)=0 \tag{1}$$

in which the whole or partial parameters are fuzzy numbers. In this paper, we dedicate to propose a HCG algorithm for solving FOP.

2. Notation and the space of fuzzy number

A fuzzy set on $\mathbb{R}^n$ is a mapping u : $\mathbb{R}^n \to [0,1]$. For each fuzzy set u, we denote its α-level set as $[u]^\alpha = \{x \in \mathbb{R}^n \,|u(x) \geq \alpha\}$ for any $\alpha \in (0.1]$. The support of u is denoted by supp(u), where supp(u) = $\{x \in \mathbb{R}^n | u(x) > 0\}$. The closure of supp(u) defines the 0-level of u, .i.e. $[u]^0 =$ cl(supp(u)), where cl(M) means the closure of the subse $M \subset \mathbb{R}^n$.

Definition 2.1. A fuzzy set u on $\mathbb{R}$ is said to be a fuzzy number, if :

(1) u is normal, i.e. there exists $x_0 \in \mathbb{R}$ such that u(x_0) $=1$;

(2) u is an upper semi-continuous function;

(3) u($\lambda x + (1-\lambda) y) \geq min\{u(x), u(y)\}, x, y \in \mathbb{R}, \lambda \in [0,1]$;

(4) $[u]^0$ is compact.

Let $\mathcal{F}_C$ denote the family of all fuzzy numbers. So, for any u $\in \mathcal{F}_C$ we have that $[u]^\alpha \in \mathcal{K}_C$ for all $\alpha \in [0,1]$, where $\mathcal{K}_C$ denote the space of all compact intervals in $\mathbb{R}$, and thus the α -levels of a fuzzy number are given by $[u]^\alpha = [\underline{u}_\alpha, \bar{u}_\alpha], \underline{u}_\alpha, \bar{u}_\alpha \in \mathbb{R}$ for all $\alpha \in [0,1]$. Triangular fuzzy numbers are a special type of fuzzy numbers which are well determined by three real number $a \leq b \leq c$ and we write $u = (a,b,c)$ and $[u]^\alpha = [a + (b-a)\alpha, c - (c-b)\alpha]$, for all $\alpha \in [0,1]$.

For fuzzy numbers u, $v \in \mathcal{F}_C$ represented by $[\underline{u}_\alpha, \bar{u}_\alpha]$ and $[\underline{v}_\alpha, \bar{v}_\alpha]$, respectively, and for real number λ, we define the addition $u+v$ and scalar multiplication λu as follows.

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}$$

$$(\lambda u)(x)=\begin{cases} u\left(\dfrac{x}{\lambda}\right), & if\ \ \lambda\neq 0. \\ 0, & if\ \ \lambda=0. \end{cases}$$

It is well known that, for every $\alpha\in[0,1]$,

$$[u+v]^{\alpha}=\left[\left(\underline{u+v}\right)_{\alpha},\left(\overline{u+v}\right)_{\alpha}\right]=\left[\underline{u}_{\alpha}+\underline{v}_{\alpha}\ ,\ \overline{u}_{\alpha}+\overline{v}_{\alpha}\right], \tag{2}$$

and

$$[\lambda u]^{\alpha}=\left[\left(\underline{\lambda u}\right)_{\alpha},\left(\overline{\lambda u}\right)_{\alpha}\right]=\left[min\{\lambda\underline{u}_{\alpha},\lambda\overline{u}_{\alpha}\},\ max\{\lambda\underline{u}_{\alpha},\lambda\overline{u}_{\alpha}\}\right]. \tag{3}$$

3. Differentiable fuzzy function

Henceforth, K denotes an open subset of $\mathbb{R}$, and $F:K\to\mathcal{F}_C$ be a fuzzy function. For each $\alpha\in[0,1]$, we define the family of interval-valued functions $F:K\to\mathcal{K}_C$, associated with F, and given by

$F_{\alpha}(x)=[F(x)]^{\alpha}$. For any $\alpha\in[0,1]$, we denote

$F_{\alpha}(x)=\left[\underline{f}_{\alpha}(x),\overline{f}_{\alpha}(x)\right]$.

Here, for each $\alpha\in[0,1]$, the endpoint functions $\underline{f}_{\alpha}$, $\overline{f}_{\alpha}$ $:K\to\mathbb{R}$ are called upper and lower function of F, respectively.

Definition 3.1. Let $K\subset\mathbb{R}$ with $F:K\to\mathcal{F}_C$ a fuzzy function and $x_0\in K$ and h be such that $x_0+h\in K$.

Then the generalized Hukuhara derivative (gH-derivative, for short) of F at x_0 is defined as

$$F'(x_0)\ =\ \lim_{h\to 0^+}\frac{1}{h}\left[F(x_0+h)\ominus_H F(x_0)\right]\ . \tag{4}$$

If $F'(x_0)\in\mathcal{F}_C$ satisfying (4) exists, we say that F is Hukuhara differentiable (H-differentiable, for short) at x_0 .

4. Necessary and sufficient optimality conditions for constrained fuzzy optimization problem

Let $T\subseteq\mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $\tilde{f},\tilde{g}_j$, for $j=1,\dots,m$ be fuzzy-valued functions defined on T. Consider the following nonlinear fuzzy optimization problem

$$\text{(FOP)}\quad \min\ \ \tilde{f}=\tilde{f}(x_1,\dots,x_n)$$
$$\text{s.t}\quad \tilde{g}_j(x)\leq\tilde{0},\ j=1,\dots,m$$

where $\tilde{0}$ is a fuzzy number defined as $\tilde{0}(r)=1$ if $r=0$ and $\tilde{0}(r)=0$ if $r\neq 0$ and its level set is $\tilde{0}_{\alpha}=\{0\}$ for $\alpha\in[0,1]$.

First we introduce the concept of convexity for fuzzy-valued functions.

Definition 4.1. Let T be a convex subset of $\mathbb{R}^n$ and $\tilde{f}$ be a fuzzy-valued function defined on T. We say that $\tilde{f}$ is convex at x^0 if:

$$\tilde{f}\left(\lambda x^0+(1-\lambda)x\right)\leq(\lambda\odot\tilde{f}\left(x^0\right)\oplus\left((1-\lambda)\odot\tilde{f}(x)\right)$$

For each $\lambda\in(0,1)$ and $x\in T$.

Proposition 4.2. $\tilde{f}:T\to F(\mathbb{R})$ is convex at x^0 if and only if $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are convex at x^0, for all $\alpha\in[0,1]$.

Kuhn-Tucker optimality conditions for (FOP) is considered as follows [6]:

Theorem 4.3. Let the fuzzy-valued objective function $\tilde{f}:T\to F(\mathbb{R})$ is convex and continuously H-differentiable, where $T\subset\mathbb{R}^n$ is open and convex. For $j=1,\dots,m$, the fuzzy-valued constraint functions $\tilde{g}_j:T\to F(\mathbb{R})$ are convex and continuously H-differentiable. Let $X=\left\{x\in T\subset\mathbb{R}^n:\tilde{g}_j(x)\leq\tilde{0},\ j=1,\dots,m\right\}$ be a feasible set of problem (FOP) and let $x^0\in X$. Suppose there is some $x\in T$ such that $\tilde{g}_j(x)\prec\tilde{0},\ j=1,\dots,m$. Then x^0 is a nondominated solution of problem (FOP) over X if and only if there exist multipliers $0\leq\mu_j\in\mathbb{R},\ j=1,\dots,m$, such that the Kuhn-Tucker first order conditions hold:

(FKT-1) $\int_0^1\nabla\tilde{f}_\alpha^L\left(x^0\right)d\alpha+\int_0^1\tilde{f}_\alpha^U\left(x^0\right)d\alpha+\sum_{j=1}^m\mu_j\nabla\tilde{g}_{j0}^U\left(x^0\right)=0;$

(FKT-2) $\mu_j\ .\tilde{g}_{j0}^U\left(x^0\right)=0$ for all $j=1,\dots,m$.

5. Hybrid Conjugate Gradient (HCG) algorithm

The parametric form of the fuzzy nonlinear Equation (1) can be written as

$$\begin{cases}\underline{F}\left(\underline{x},\overline{x},r\right)=0,\\ \overline{F}\left(\underline{x},\overline{x},r\right)=0,\end{cases}\qquad\forall r\in[0,1].\tag{5}$$

Define the function $G_r:\mathbb{R}^2\to\mathbb{R}$ as follows:

$$G_r(x)=G_r\left(\underline{x},\overline{x}\right)=\underline{F}^2\left(\underline{x},\overline{x},r\right)+\overline{F}^2\left(\underline{x},\overline{x},r\right),\tag{6}$$

where $r\in[0,1]$.

Then the problem (5) can be transformed into the following problem:

$$\min_{x\in u}G_r(x)\tag{7}$$

Obviously, for the same parameter r, the solution $(\underline{x}^*,\overline{x}^*)$, which satisfies the object function $G_r\left(\underline{x}^*,\overline{x}^*\right)=0$, is also one of (5).

Remark 5.1.
$G_r\left(\underline{x},\overline{x}\right)$ in (6) is different with the following**,** $G_r\left(\underline{x},\overline{x}\right)$ definted in [2]:

$$G_r\left(\underline{x},\overline{x}\right)=\left[\underline{F}\left(\underline{x},\overline{x},r\right)+\overline{F}\left(\underline{x},\overline{x},r\right)\right]^2 \tag{8}$$

It is clear that the function $G_r\left(\underline{x},\overline{x}\right)$ in (6) is more reasonable than (8).
Definition 5.2. [2]
Let . . The gradient of G_r at x is denoted by $\nabla G_r\left(x\right)$ and defined by

$$\nabla G_r\left(x\right)=\left(\frac{\partial G_r}{\partial \underline{x}},\frac{\partial G_r}{\partial \overline{x}}\right)^r \tag{9}$$

Now, we apply a hybrid conjugate gradient algorithm to solve the problem (7).
Main steps of HCG algorithm 5.3.
Step 1. Give an initial solution $x_0\left(r\right)\in u$ and a tolerance error parameter $\xi\geq 0$.
Let $d_1=-\nabla G_r\left(x_0\right)$, $k:=1.$
Step 2. If $\nabla G_r\left(x\right)\leq\xi$, then stop; or obtain step length α_k by using exact line search

$$\min_{\alpha\geq 0} G_r(x_k\left(r\right)+\alpha d_k), \tag{10}$$

and update iterative solution by $x_{k+1}\left(r\right)=x_k\left(r\right)+\alpha_k d_k$,
Step 3. Get parameter β_{k+1} by $\beta_{k+1}=max\left\{0,min\left\{\beta_{k+1}^{FR},\beta_{k+1}^{PRP}\right\}\right\}$, (11)

where, $$\beta_{k+1}^{FR}=\frac{\nabla G_r\left(x_{k+1}\right)^2}{\nabla G_r\left(x_k\right)^2}, \tag{12}$$

$$\beta_{k+1}^{PRP}=\frac{\nabla G_r\left(x_{k+1}\right)^T\left(\nabla G_r\left(x_{k+1}\right)-\nabla G_r\left(x_k\right)\right)}{\nabla G_r\left(x_k\right)^2}. \tag{13}$$

Step 4. Define search direction by

$$d_{k+1}=-\nabla G_r\left(x_{k+1}\right)+\beta_{k+1}d_k \tag{14}$$

Let $k=k+1$, return step 2 .

6. Numerical example

We consider here a fuzzy optimization problem having fuzzy-valued objective function [6]

Example 6.1.

$$\min\ \tilde{f}\left(x_1,x_2\right)=\left(\tilde{a}\odot x_1^2\right)\oplus\left(\tilde{b}\odot x_2^2\right)$$

$$\text{s.t}\quad g\left(x_1,x_2\right)=\left(x_1-2\right)^2+\left(x_2-2\right)^2\leq 1,$$

where $\tilde{a}=\left(1,2,3\right)$ and $\tilde{b}=\left(0,1,2\right)$ are triangular fuzzy numbers defined on $\mathbb{R}$ as

$$\tilde{a}\left(r\right)=\begin{cases}\left(r-1\right), if\ 1\leq r\leq 2,\\ \left(3-r\right), if\ 2<r\leq 3,\\ 0, \quad otherwise\end{cases} \quad \text{and} \quad \tilde{b}\left(r\right)=\begin{cases}r, & if\ 0\leq r\leq 1,\\ \left(2-r\right), & if\ 1<r\leq 2,\\ 0, & otherwise\end{cases}$$

Now, we obtain

$\tilde{f}_\alpha^L(x_1,x_2)=(1+\alpha)x_1^2+\alpha x_2^2$ and $\tilde{f}_\alpha^U(x_1,x_2)=(3-\alpha)x_1^2+(2-\alpha)x_2^2$ for $\alpha\in[0,1]$.

Also,

$$\nabla\tilde{f}_\alpha^L(x_1,x_2)=\begin{pmatrix}2(\alpha+1)x_1\\2\alpha x_2\end{pmatrix},\quad \nabla f_\alpha^U(x_1,x_2)=\begin{pmatrix}2(3-\alpha)x_1\\2(2-\alpha)x_2\end{pmatrix} \text{ and, } \nabla g(x_1,x_2)=\begin{pmatrix}2(x_1-2)\\2(x_2-2)\end{pmatrix},$$

Therefore, we have

$$\int_0^1\nabla\tilde{f}_\alpha^L(x_1,x_2)\,d\alpha=\begin{pmatrix}3x_1\\x_2\end{pmatrix},\quad \int_0^1\nabla\tilde{f}_\alpha^U(x_1,x_2)\,d\alpha=\begin{pmatrix}5x_1\\3x_2\end{pmatrix}.$$

From Theorem 2, we have the following Kuhn-Tucker conditions

(FKT-1) $8x_1+2\mu(x_1-2)=0,\ 4x_2+2\mu(x_2-2)=0$,

(FKT-2) $\mu\left((x_1-2)^2+(x_2-2)^2-1\right)=0$.

Solving these equations, we get the solution $(x_1,x_2)=(6/5,3/2)$ and $\mu=6$. By Theorem2, we say that $(x_1^*,x_2^*)=(6/5,3/2)$ is nondominated solution for given problem. Also the minimum value of objective function is $\tilde{f}_{min}=(1.44,5.13,8.82)$ and we can find its defuzzified value 5.13 by using center of area method.

7. Conclusion

In this paper, we improved the objective function of FO problem in [2], and proposed HCG algorithm for solving FOP and analyze its convergence. Initially, we wrote a fuzzy constrained optimization in a parametric form and then solve it by the steepest descent method. Finally, a numerical example used to illustrate the proposed method.

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