International Journal of Applied Operational Research
Vol. 9, No. 4, pp. 49-54, Autumn 2019

Journal homepage: ijorlu.liau.ac.ir

Proof of a conjecture of Yuan and Zontini on preconditioned methods for M-matrices

S. A. Edalatpanah*, S. E. Najafi

Received: 12 May 2019 **Accepted:** 24 August 2019

Abstract Systems of linear equations are ubiquitous in science and engineering, and iterative methods are indispensable for the numerical treatment of such systems. When we apprehend what properties of the coefficient matrix account for the rate of convergence, we may multiply the original system by some nonsingular matrix, called a preconditioner, so that the new coefficient matrix possesses better properties. Recently, some scholars presented several preconditioners and based on numerical tests proposed some conjectures for preconditioned iterative methods. In this paper, we prove one conjecture on the preconditioned Gauss–Seidel iterative method for solving linear systems whose coefficient matrix is an M-matrix.

Keywords: Iterative Methods, Preconditioning, Comparison Theorems, Spectral Radius, Conjecture, Gauss–Seidel, M-Matrix.

1 Introduction

Let us consider the following linear systems:

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$ and $b, x \in R^n$. For the simplicity, in this paper we shall assume that $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and upper triangular parts of A respectively. This problem is ubiquitous in mathematical modeling and scientific computing, and occurs in a wide variety of areas including numerical differential equation, economics models, design and computer analysis of circuits, power system networks, chemical engineering processes, physical and biological sciences. Iterative methods are indispensable for the numerical treatment of such systems, and numerous iterative methods are available to find a solution for linear systems; see [1-10] and the references therein.

Large families of these iteration methods for solving Eq. (1) take the splitting form. For any splitting A=M-N, where M is nonsingular, the iterative method for solving linear systems of Eq. (1) is:

*Corresponding Author. (✉)
E-mail: saedalatpanah@gmail.com (S.A. Edalatpanah)

S. A. Edalatpanah
Assistance Professor, Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran.

S. E. Najafi
Assistant Professor, Department of Industrial Engineering, Islamic Azad University, Central Tehran Branch, Tehran 14676-86831, Iran.

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \qquad i = 0,1,\ldots \tag{2}$$

This iterative process converges to the unique solution $x = A^{-1}b$ for initial vector value $x^0 \in R^n$ if and only if $\rho(M^{-1}N) < 1$, where $T = M^{-1}N$ is called the iteration matrix and $\rho(T)$ denotes the spectral radius of T . The effective method to decrease the spectral radius is to preconditioning technique. A preconditioner is defined as an auxiliary approximate solver which will be combined with an iterative method. Therefore, a basic idea of preconditioned iterative methods is to transform Eq.(1) into the preconditioned form *PAx=Pb.*

In literature, various authors have suggested different models of *(I+S)-type* preconditioner for the above mentioned problem; see [1, 9-10, 11-17]. For example, Milaszewicz [1], presented the preconditioner *(I+S)*, where the elements of the first column below the diagonal of A eliminate.

Gunawardena, Jain and Snyder in [11] proposed a modification of Jacobi and Gauss-Seidel methods and reported that the convergence rate of the Gauss–Seidel method using the following preconditioning matrix is superior to that of the standard Gauss–Seidel method;

$$P = I + S,$$

and

$$S = (s_{i,j}) = \begin{cases} -a_{i,j}, & for \quad j = i+1, i = 1,2,\ldots,n-1 \\ 0, & for \quad otherwise. \end{cases}$$

Inspiring from the same idea, Kohno *et al*. [12] propose an extended modification of Jacobi and Gauss-Seidel methods. Usui et al. [13] proposed to adopt:

$$P = I + U,$$

as the preconditioned matrix, where U is strictly upper triangular of matrix *A*. They obtained excellent convergence rate compared with that by other iterative methods.

Evans et al. [14] proposed the preconditioner $P = I + R_1$, where

$$R_1 = \begin{cases} -a_{i,j}, & for \quad i = n, j = 1 \\ 0, & for \quad otherwise. \end{cases}$$

In 2001, Niki et al. [15] proposed the preconditioner $P = I + S + R,$, where

$$R = \begin{cases} -a_{i,j}, & for \quad i = n, j = 1,2,\ldots,n-1 \\ 0, & for \quad otherwise. \end{cases}$$

Saberi Najafi and Edalatpanah in [16] established;

$$\tilde{P} = I + S_{\min},$$

where for $i = 1:n-1$ and $j > i$:

$$S_{\min} = \begin{cases} 0, & if \quad j \in Q_i. \\ -a_{i,j}, & otherwise. \end{cases}$$

and

$$Q_i = \left\{ j \;\middle|\; j < i \,\&\, |a_{i,j}| = \min_k |a_{i,k}| \right\} \qquad for \quad i < n-1.$$

Some scholars also used these preconditioner models for algebraic systems and complementarity problems [18-21]. Furthermore, some other researchers have considered different models in the literature [22-29].

Yuan and Zontini [17], inspiring from the above models, present a new preconditioner for the Gauss–Seidel method on this class. These authors also established several comparison theorems for the proposed method with several preconditioners. However, based on numerical tests, they proposed some conjectures for preconditioned Gauss-Seidel iterative method. In this paper, based on nonnegative matrix analysis, we prove the conjecture.

2 Preliminaries

In this section, we give some basic notations and preliminary results which are essential tools for describing our main results. Our notations and definitions in this paper are standard, and for details, we refer to [2-3, 9].

Definition 2.1. A real $n\times n$ matrix $A=(a_{ij})$ is called

(a) a *Z-matrix* if and only if for any $i \neq j, a_{ij} \leq 0$;

(b) an *L-matrix* , if it is a *Z-matrix* and $a_{ii}>0$;

(c) an *M-matrix* if and only if it is a *Z-matrix* , nonsingular and $A^{-1} \geq 0$.

Definition 2.2. Let A be a real matrix. The splitting $A=M-N$ is called

(a) convergent if $\rho(M^{-1}N)<1$,

(b) regular if $M^{-1} \geq 0$ and $N \geq 0$,

(c) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Clearly, a regular splitting is weak regular.

Lemma 2.1. Let $A = M - N$ is regular or weak regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.

Lemma 2.2. Let A, B are *Z-matrix* and A is an *M-matrix* , if $A\leq B$ then B is also an *M-matrix* .

Lemma 2.3. Let A be a nonnegative matrix. Then

(i) If $\alpha x \leq Ax$ for some nonnegative vector x, then $\alpha \leq \rho(A)$.

(ii) If $\alpha x \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and $0 \neq \alpha x \leq Ax \leq \beta x, \alpha x \neq Ax, Ax \neq \beta x$ for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

3. Main Results

Based on [17], we assume that is A is an M-matrix. Furthermore, we consider the following preconditioners:

$$\begin{cases} P_{S_1} = I + S + R_1, \\ P_{SR} = I + S + R, \\ P_U = I + U, \\ P_{RU} = I + R + U. \end{cases}$$

Their corresponding preconditioned matrices are respectively:

$$\begin{cases} A_{S_1} = P_{S_1} A, \\ A_{SR} = P_{SR} A \\ A_U = P_U A, \\ A_{RU} = P_{RU} A. \end{cases}$$

Hence, the splittings of the Gauss–Seidel method are respectively:

$$\begin{cases} A = M - N: \quad M = I - L; N = U, \\ A_{S_1} = M_{S_1} - N_{S_1}: \quad M_{S_1} = I - L + R_1 - SL - R_1 U; N_{S_1} = U - S + SU, \\ A_{SR} = M_{SR} - N_{SR}: \quad M_{SR} = I - L + R - RL - SL - RU; N_{SR} = U - S + SU, \\ A_U = M_U - N_U: \quad M_U = I - L - E_0; N_U = U^2 + F_0, \\ A_{RU} = M_{RU} - N_{RU}: \quad M_{RU} = I - L + R - RL - RU - E_0; N_{RU} = U^2 + F_0. \end{cases}$$

where E_0, F_0 denote the lower triangular and upper triangular parts of $UL = E_0 + F_0$, respectively.

Theorem 3.1. (Conjecture 6.1)Let A be an *M-matrix*. Then we have;

$$\rho(M_{SR}^{-1} N_{SR}) \le \rho(M_{S_1}^{-1} N_{S_1}).$$

Proof. We first consider the case where A_{SR} is an irreducible M-matrix. Hence, $A_{SR} = M_{SR} - N_{SR}$ is regular spitting, then by [3; Theorem 2.7], there exists a positive vector x such that $(M_{SR}{}^{-1} N_{SR})x = \rho(M_{SR}{}^{-1} N_{SR})x.$
Furthermore, since $N_{SR} \ge 0$ we have $N_{SR} x \ge 0$. And so,

$$M_{SR} x = \frac{1}{\rho(M_{SR}{}^{-1} N_{SR})} N_{SR} x \ge 0.$$

Therefore,

$$A_{SR} x = M_{SR}(I - M_{SR}{}^{-1} N_{SR})x = \frac{1 - \rho(M_{SR}{}^{-1} N_{SR})}{\rho(M_{SR}{}^{-1} N_{SR})} N_{SR} x \ge 0.$$

Furthermore, we know that $A_{SR} x = (I + S + R)Ax \ge 0$ and $(I + S + R) \ge 0$, therefore $Ax \ge 0$.
Now we have;

$$
\begin{aligned}
&A_{SR}x = (I+S+R)Ax \\
&= (I+S+R)Ax + (R_1 - R_1)Ax \\
&= (I+S+R_1)Ax + (R-R_1)Ax \geq (I+S+R_1)Ax \geq 0 \\
&\Rightarrow A_{SR}x \geq A_{S_1}x.
\end{aligned}
$$

On the other hand, by definition, $\bar{S}_{\bar{\alpha}_q}U$ and $(\bar{S}_{\alpha_q} - \bar{S}_{\bar{\alpha}_q})U$ are nonnegative, so;

$$
\begin{aligned}
\bar{M}_{\alpha_q} &= I - (\bar{S}_{\bar{\alpha}_q}U)_D - ([\bar{S}_{\alpha_q} - \bar{S}_{\bar{\alpha}_q}]U)_D \\
&\quad - r\{L + (\bar{S}_{\bar{\alpha}_q}U)_L + ([\bar{S}_{\alpha_q} - \bar{S}_{\bar{\alpha}_q}]U)_L + \bar{S}_{\alpha_q}\} \\
&\quad \leq I - L + R_1 - SL - R_1U = M_{S_1},
\end{aligned}
$$

where $(\kappa)_D$ and $(\kappa)_L$ denote the diagonal and strictly lower triangular parts of κ, respectively. In view of that both $\bar{M}_{\alpha_q}$ and $\bar{M}_{\bar{\alpha}_q}$ are *M-matrices*, $(\bar{M}_{\alpha_q})^{-1} \geq (\bar{M}_{\bar{\alpha}_q})^{-1}$.

Then we have;

$$
\begin{aligned}
&\rho(\bar{M}_{\alpha_q}{}^{-1}\bar{N}_{\alpha_q})x = \bar{M}_{\alpha_q}{}^{-1}\bar{N}_{\alpha_q}x = x - \bar{M}_{\alpha_q}{}^{-1}\bar{A}_{\alpha_q}x \\
&\leq x - \bar{M}_{\alpha_q}{}^{-1}\bar{A}_{\bar{\alpha}_q}x \leq x - \bar{M}_{\bar{\alpha}_q}{}^{-1}\bar{A}_{\bar{\alpha}_q}x \\
&= (I - \bar{M}_{\bar{\alpha}_q}{}^{-1}\bar{A}_{\bar{\alpha}_q})x = \bar{M}_{\bar{\alpha}_q}{}^{-1}\bar{A}_{\bar{\alpha}_q}x.
\end{aligned}
$$

Therefore by Lemma 2.3 the proof is completed.■

4 .Conclusion

From the results, it may be concluded that preconditioners are effective to accelerate convergence of the iterative methods. Furthermore, we prove one conjecture on the preconditioned Gauss–Seidel iterative method for solving linear systems whose coefficient matrix is an M-matrix.

References

1. Milaszewicz, J. P. (1987). Improving Jacobi and Gauss-Seidel iterations. Linear Algebra and Its Applications, 93, 161-170.
2. Hageman, L. A., & Young, D. M. (2012). Applied iterative methods. Courier Corporation.
3. Varga, R. S. (1962). Iterative analysis. Prentice Hall, Englewood Cliffs, NJ.
4. Saad, Y. (2003). Iterative methods for sparse linear systems (Vol. 82). siam.
5. Saberi Najafi, H., & Edalatpanah, S. A. (2011). Fast iterative method-FIM. Application to the convection–diffusion equation. Journal of Information and Computing Science, 6(4), 303-313.
6. Najafi, H. S., & Edalatpanad, S. A. (2013). On application of Liao's method for system of linear equations. Ain Shams Engineering Journal, 4, 501-505.
7. Najafi, H. S., & Edalatpanah, S. A. (2013). Two stage mixed-type splitting iterative methods with applications. Journal of Taibah University for Science, 7(2), 35-43.
8. Najafi, H. S., & Edalatpanah, S. A. (2014). A new modified SSOR iteration method for solving augmented linear systems. International Journal of Computer Mathematics, 91(3), 539-552.
9. Edalatpanah, S. A. (2018). On the modified methods for irreducible linear systems with L-matrices. Bulletin of Computational Applied Mathematics, 6(1),119-128.

10. Najafi, H. S., Edalatpanah, S. A., & Refahisheikhani, A. H. (2018). An analytical method as a preconditioning modeling for systems of linear equations. Computational and Applied Mathematics, 37(2), 922-931.
11. Gunawardena, A. D., Jain, S. K., & Snyder, L. (1991). Modified iterative methods for consistent linear systems. Linear Algebra and Its Applications, 154, 123-143.
12. Kohno, T., Kotakemori, H., Niki, H., & Usui, M. (1997). Improving the modified Gauss-Seidel method for Z-matrices. Linear Algebra and its Applications, 267, 113-123.
13. Usui, M., Niki, H., & Kohno, T. (1994). Adaptive Gauss-Seidel method for linear systems. International Journal of Computer Mathematics, 51(1-2), 119-125.
14. Evans, D. J., Martins, M. M., & Trigo, M. E. (2001). The AOR iterative method for new preconditioned linear systems. Journal of computational and applied mathematics, 132(2), 461-466.
15. Niki, H., Kohno, T., & Morimoto, M. (2008). The preconditioned Gauss–Seidel method faster than the SOR method. Journal of Computational and Applied Mathematics, 219(1), 59-71.
16. Najafi, H. S., & Edalatpanah, S. A. (2012). New model for preconditioning techniques with application to the boundary value problems. J. Adv. Res. Computer Engng: An Intern. J, 6, 107-114.
17. Yuan, J. Y., & Zontini, D. D. (2012). Comparison theorems of preconditioned Gauss–Seidel methods for M-matrices. Applied Mathematics and Computation, 219(4), 1947-1957.
18. Najafi, H. S., & Edalatpanah, S. A. (2012). New model for preconditioning techniques with application to the boundary value problems. J. Adv. Res. Computer Engng: An Intern. J, 6, 107-114.
19. Najafi, H. S., & Edalatpanah, S. A. (2013). Iterative methods with analytical preconditioning technique to linear complementarity problems: application to obstacle problems. RAIRO-Operations Research, 47(1), 59-71.
20. Edalatpanah, S. A. (2018). On the preconditioned projective iterative methods for the linear complementarity problem. RAIRO-Operations Research. doi:10.1051/ro/2019002
21. Mao, X., Wang, X., Edalatpanah, S. A., & Fallah, M. (2019). The Monomial Preconditioned SSOR Method for Linear Complementarity Problem. IEEE Access, 7, 73649-73655.
22. Najafi, H. S., Edalatpanah, S. A., & Sheikhani, A. R. (2014). Convergence analysis of modified iterative methods to solve linear systems. Mediterranean journal of mathematics, 11(3), 1019-1032.
23. Najafi, H. S., & Edalatpanah, S. A. (2015). A new family of (I+ S)-type preconditioner with some applications. Computational and Applied Mathematics, 34(3), 917-931.
24. Li, C. X. (2017). A preconditioned AOR iterative method for the absolute value equations. International Journal of Computational Methods, 14(02), 1750016.
25. Huang, Z. G., Wang, L. G., Xu, Z., & Cui, J. J. (2018). Some new preconditioned generalized AOR methods for solving weighted linear least squares problems. Computational and Applied Mathematics, 37(1), 415-438.
26. Tian, Z., Tian, M., Zhang, Y., & Wen, P. (2018). An iteration method for solving the linear system Ax= b. Computers & Mathematics with Applications, 75(8), 2710-2722.
27. Li, W., Liu, D., & Vong, S. W. (2018). Comparison results for splitting iterations for solving multi-linear systems. Applied Numerical Mathematics, 134, 105-121.
28. Sunarto, A., & Sulaiman, J. (2019, July). Preconditioned SOR Method to Solve Time-Fractional Diffusion Equations. In Journal of Physics: Conference Series (Vol. 1179, No. 1, p. 012020). IOP Publishing.
29. Bruaset, A. M. (2018). A survey of preconditioned iterative methods. Routledge.