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A new approach to marginal rate analysis in DEA with a focus on maintaining profitability

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Abstract Analyzing the effects of marginal changes in input and output variables—referred to as throughputs on economic outcomes is a critical concern in both economic theory and practice. Marginal Rates (MR) play a key role in assessing the sensitivity of economic systems to such variations. This study enhances a recently introduced Data Envelopment Analysis (DEA) model, originally developed for profitability assessment, by incorporating marginal rate analysis within its framework. A binary-variable-based methodology is proposed to examine the marginal rates, enabling the simultaneous evaluation of how minor variations in one throughput affect others. By leveraging the concept of profitability in DEA, the proposed approach provides a comprehensive explanation of how decision-making units (DMUs) attain and sustain profitability. The proposed Mixed MR model offers a robust analytical tool for examining the interdependencies between performance indicators within efficient units. An empirical application involving branches of an Iranian bank demonstrates the effectiveness of the method in revealing the influence of individual indicators on one another in efficient operational contexts.

Keyword: Data Envelopment Analysis (DEA), Marginal Rates, Profitability Analysis, Decision-Making Units (DMUs), Bank Branch Performance, Managerial Decision-Making.

1 Introduction

Given the critical role that companies play in fostering economic development, identifying effective strategies to enhance their performance is of paramount importance. A fundamental objective in production theory is the development and optimization of decision-making units (DMUs), as sustaining efficiency and improving productivity are essential for success in today's competitive business environment.

Among the various tools available for performance evaluation, Data Envelopment Analysis (DEA) has emerged as a prominent and widely adopted method. Originally introduced by Charnes et al. [1] under the assumption of constant returns to scale (the CCR model), DEA has gained traction both in theoretical research and practical applications. As a non-parametric method, DEA facilitates the assessment of organizational performance involving multiple

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inputs and outputs, enabling comparative analysis across peer units to identify drivers of efficiency and profitability.

The flexibility and applicability of DEA have led to its widespread use in diverse sectors such as finance, manufacturing, healthcare, transportation, and especially banking. It empowers decision-makers to benchmark performance, optimize resource utilization, and identify opportunities for strategic improvement. Extensive research has been devoted to advancing the DEA methodology, both conceptually and computationally. Pioneering works by scholars such as Banker et al. [2, 3], Seiford and Thrall [4], Charnes et al. [5], Seiford [6], Bessent et al. [7], Cooper et al. [8], and Emrouznejad et al. [9] have significantly contributed to the development and application of DEA across various disciplines.

A critical aspect of performance analysis in economics and management is the evaluation of marginal changes in inputs and outputs, and their cascading effects on other variables. Due to the interdependent nature of production processes, altering one input or output often influences others—a phenomenon that must be carefully accounted for. This interrelationship is formally captured by the concept of Marginal Rates (MR) in the DEA context, typically assessed through partial derivatives.

Understanding these marginal effects is essential for gaining deeper insights into system behavior and informing resource allocation and policy decisions. In response, researchers have proposed a variety of methodologies to model and manage such effects within the DEA framework. For instance, Asmild et al. [10] introduced a revised version of Rosen et al.'s [11] method, presenting a generalized framework for evaluating marginal rates on DEA frontiers. Similarly, Smith et al. [12] investigated the indirect and inseparable impacts of input and output changes, highlighting their significance in performance assessment.

Further contributions to this field include the works of Williams et al. [13], Ouellette et al. [14], Sueyoshi et al. [15, 16], Wang [17], Gunawardana [18], Bozorgi et al. [19], and Jalalet et al. [20], each offering valuable insights into the application of marginal analysis within DEA. Notably, Amirteimoori, et al. [21] proposed a marginal rate model based on stochastic DEA using chance-constrained programming, addressing environments characterized by data uncertainty. In another study, Wu et al. [22] examined the marginal cost of reducing carbon dioxide emissions, emphasizing that such reductions are not cost-free and proposing methodologies for estimating the associated economic trade-offs. These developments underline the growing interest in marginal rate analysis within the DEA framework and its relevance to contemporary decision-making challenges.

The remainder of this study is organized as follows. Section 2 introduces the concept of profitability analysis and marginal rates (MRs) within the framework of Data Envelopment Analysis (DEA). Section 3 presents a novel Mixed Marginal Rate (MMR) model. Section 4 demonstrates the applicability of the proposed model through an empirical study involving branches of an Iranian bank. Finally, Section 5 concludes the paper by summarizing the key findings and their implications.

2 Analysis of Profitability and Marginal Rates

This section delves into two fundamental concepts within the Data Envelopment Analysis (DEA) framework profitability and marginal rates (MRs) both of which are pivotal for evaluating and enhancing the performance of decision-making units (DMUs).

Profitability serves as a key performance indicator, reflecting the capacity of DMUs to transform inputs into financial gains. Analyzing profitability not only facilitates the assessment

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of how efficiently resources are utilized to generate revenue, but also helps in identifying inefficiencies and potential areas for strategic improvement. In competitive and resource-constrained environments, maximizing profitability is essential for ensuring long-term sustainability and value creation.

Complementing this analysis, marginal rates offer a nuanced understanding of the sensitivity of outputs to variations in inputs. Specifically, MRs quantify the incremental change in output resulting from a marginal change in a given input, holding other factors constant. This perspective is crucial in DEA applications, as it enables the assessment of the responsiveness of production processes, thereby guiding decision-makers in resource allocation and operational adjustments.

In the context of DEA, marginal rate analysis provides a deeper insight into the structure of the efficient frontier, revealing how small shifts in resource utilization can influence overall performance. The ability to analyze these micro-level changes is particularly valuable for identifying leverage points within the production system where minor adjustments may yield disproportionately beneficial outcomes.

The subsections that follow present a structured examination of these two concepts. First, the role and assessment of profitability in the DEA framework are discussed. Then, a detailed analysis of marginal rates is provided, including their formulation, interpretation, and implications for decision-making and performance optimization.

2.1 Profitability Analysis

In many industries, profitability serves as a reliable metric for evaluating the performance of production activities. As a result, it is widely used internationally as a comparable and valuable criterion for assessing the performance of units. Calculating profitability offers valuable insights into the economic condition of companies and helps identify appropriate trends throughout the business cycle.

Let's explore the theory behind the profitability model. Consider a technology represented by $T = \{(\mathrm{x}, \mathrm{y}) \in \mathrm{R}^{m+s} : x \ \ can\ produce\ y\}$, where x is an input vector $(i = 1,\ldots,m)$ and y is an output vector $(r = 1,\ldots,s)$.We assume that T is represented by the following production possibility set:

$$T = \{(\overline{x}, \overline{y}) : \overline{x} \geq \sum_{j=1}^{n} \lambda_j x_{ij}, \overline{y} \leq \sum_{j=1}^{n} \lambda_j y_{rj}, \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0\}.$$

T is strongly free disposable, convex set satisfying constant returns to scale (VRS).

In a two-dimensional space, the measure of technical efficiency is the ratio of productivity (defined as the output-to-input ratio) at the observed point to that at the target point on the frontier of $T : (\frac{py}{cx}) \Big/ (\frac{py^*}{cx^*})$. Let an observed vector (*x, y*) and the target vector (*x*, y**) on the efficient frontier.

To evaluate profitability, We consider a common unit price vector $p = (p_1,\ldots,p_s)$ for the output $y = (y_1,\ldots,y_s)$ and a unit cost vector $c = (c_1,\ldots,c_m)$ for the input $x = (x_1,\ldots,x_m)$. The problem of identifying the profit-maximizing input-output combination within the production possibility set can be formulated as the following fractional programming model**:**

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$$W = \frac{py^*}{cx^*} = Max \ \frac{p\bar{y}}{c\bar{x}}$$
$$\begin{aligned} s.t. \quad & \bar{x} = \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} && i = 1,\dots,m; \\ & \bar{y} = \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && r = 1,\dots,s; \\ & \sum_{j=1}^{n} \lambda_j = 1; \\ & \lambda_j \ge 0 && j = 1,\dots,n. \end{aligned} \tag{1}$$

Let (x^*, y^*) be the optimal solution to problem (1) and let (x_o, y_o) be the vector of observed values for DMU_o , Then profit efficiency can be expressed as:

$$E_p = \left(\frac{p_o y_o}{c_o x_o}\right) \Big/ \left(\frac{p_o y^*}{c_o x^*}\right).$$

Since $\frac{p_o y_o}{c_o x_o} \le \frac{p_o y^*}{c_o x^*}$ we have $0 < E_p \le 1$ and DMU_o is profit efficient if and only if $E_p = 1$.

To solve the fractional model (1) using linear programming techniques, we apply the **Charnes-Cooper transformation.** Let $t > 0$ and define transformed variables $\hat{x} = t\bar{x}, \hat{y} = t\bar{y}, \hat{\lambda} = t\lambda$ accordingly. By multiplying all terms in the fractional model by t, we obtain the equivalent linear programming (LP) formulation:

$$W = Max \ p\hat{y}$$
$$\begin{aligned} s.t. \quad & \hat{x} = \sum_{j=1}^{n} \hat{\lambda}_j x_{ij} \le t x_{io} && i = 1,\dots,m; \\ & \hat{y} = \sum_{j=1}^{n} \hat{\lambda}_j y_{rj} \ge t y_{ro} && r = 1,\dots,s; \\ & \sum_{j=1}^{n} \hat{\lambda}_j = t, \\ & \hat{\lambda}_j \ge 0 && j = 1,\dots,n. \end{aligned} \tag{2}$$

Suppose an optimal solution of this linear program problem be $(t^*, \hat{x}^*, \hat{y}^*, \hat{\lambda}^*)$. Due to the $t > 0$, we can obtain an optimal solution to problem (1) from $x^* = \frac{\hat{x}}{t^*}, y^* = \frac{\hat{y}}{t^*}, \lambda^* = \frac{\hat{\lambda}}{t^*}$.

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It is clear that the differences between x^* and x_o, as well as between y^* and y_o, represent directions for managerial improvement, which are analyzed through the constraint in equation (1).

2.2 Marginal Rates

In microeconomic theory, particularly in production theory, the analysis of how changes in one specific throughput (input or output) affect another is a central topic. These trade-offs are formally referred to as marginal rates. Mathematically, marginal rates are represented as partial derivatives along the production frontier. The study of the impact of homogeneous throughputs in terms of input or output is referred to as substitution marginal rates, while non-homogeneous throughputs are referred to as marginal rates of transformation. The examination of such trade-offs is closely tied to the analysis of technology characteristics and the production frontier.

However, within the Data Envelopment Analysis (DEA) framework, the production frontier is piecewise linear, making the calculation of unique partial derivatives at frontier points infeasible. To address this limitation, the concepts of right-hand and left-hand marginal rates are used. These are defined as directional partial derivatives:

- The right-hand marginal rate corresponds to a small increase in the throughput of interest, and the partial derivative is evaluated from the right.
- The left-hand marginal rate reflects a small decrease in the throughput, with the derivative taken from the left.

The foundational work of Rosen et al. [1998] formally established the relationship between marginal rates and partial derivatives.

Consider a general setting where each DMU_j is characterized by a throughput vector $z_j = (-x_j, y_j)^t$, where $x_j = (x_{1j}, ..., x_{mj})$ and $y_j = (y_{1j}, ..., y_{sj})$.

Definition 1. Let $z_j = (-x_j, y_j)^t$ be a point on the production frontier. The marginal rate of substitution of the *j-th* throughput with respect to the k-th throughput at point z_o is defined as:

$$MR_{ij}^{+}(\mathbf{z}_o) \equiv \frac{\partial z_i}{\partial z_j}\Big|_{\mathbf{z}_o^+} == \lim_{h \to 0^+} \frac{z_i\left(z_{1o}, \ldots, z_{jo}+h, \ldots, z_{(m+s)o}\right) - z_i\left(z_{1o}, \ldots, z_{jo}, \ldots, z_{(m+s)o}\right)}{h}$$

$$MR_{ij}^{-}(\mathbf{z}_o) \equiv \frac{\partial z_i}{\partial z_j}\Big|_{\mathbf{z}_o^-} == \lim_{h \to 0^-} \frac{z_i\left(z_{1o}, \ldots, z_{jo}-h, \ldots, z_{(m+s)o}\right) - z_i\left(z_{1o}, \ldots, z_{jo}, \ldots, z_{(m+s)o}\right)}{h}$$

Thus, the right and left partial derivatives at a particular point represent the right-side and left-side MRs respectively.

Asmild et al [2006] proposed a four-step procedure to calculate the marginal rates of substitution of the *jth* throughput to the *kth* throughput, at the frontier point z_o. The MRS computing process is as follows:

i. Choose a small increment h for the k-th throughput.

ii. Solve the following LP problem and obtain the value of z_{jo}^*:

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$$Max \quad z^*_{jo}$$

$$st. \quad (z_{1o},\dots,z_{ko},\dots,z^*_{jo},\dots,z_{m+s,o}) \in T$$

iii.Calculate the marginal rate of substitution from right as follows:

$$MR^+_{jk}(z_o) = \frac{z^*_{jo} - z_{jo}}{h}$$

iv. Repeat phrase (ii) and (iii) for –*h* to get the marginal rate of substitution from the left.

As we can see, in the process outlined by Asmild et al., changes in a selected throughput, as chosen by the manager, are observed relative to changes in other throughputs.

In the next section, we introduce a novel marginal rate model and apply it to units that are efficient based on model (2). Accordingly, an appropriate output to maintain unit profitability is determined. This approach aids in better identifying specific output production processes.

3 Mixed Marginal Rates

In this section, we introduce a mixed marginal rate (MMR) model to evaluate the effect of a partial change in one throughput on other throughputs in a single step. The MMR model provides a systematic approach to determine the required adjustments in one throughput in order to maintain production feasibility and technical efficiency, given a marginal change in another throughput. This is particularly important when assessing the behavior of efficient and profitable Decision-Making Units (DMUs). Suppose *I* denote the set of input indices, with $i \in I$, *O* denote the set of outputs indices, with $r \in O$, *J* denote the set of *DMUs,* with $j \in J$ .Also Observations x_{ij} and y_{rj} are the *ith* input level and the *j*th output level of DMU_j respectively. Also x_{ko} shows the input of the *Kth* DMU_j, The marginal rate of profitable efficient units should be calculated in relation to its changes and y_t is the decision variable representing the maximum absolute level and best of output *tth* DMU_j for keep profitability that choose by model. Index $t \in O$ is used for one index and is an alias of *r*. As well λ_j be the decision variable referring to the intensity weights representing the convex combination between DMUs.

The MMR model aims to determine how much a selected output (y_t) must change, given a marginal variation in an input, such that the resulting combination remains within the efficient production possibility set and maintains profitability. Because only one output is to be selected for adjustment, the model introduces a binary constraint (δ_t) that ensures exactly one output is selected. Accordingly, the proposed model is defined as follows:

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$$Z = Max \; \frac{\sum_{t=1}^{s} p_t \delta_t y_t + \sum_{t=1}^{s} \delta_t \sum_{\substack{r=1 \\ r \neq t}}^{s} p_r y_{ro}}{\sum_{\substack{i=1 \\ i \neq k}}^{m} c_i x_{ij} + (x_{ko} + h)}$$

$$\begin{aligned}
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{kj} = x_{ko} + h \quad ; \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \qquad i = 1,\ldots,m \quad , i \neq k \\
& \sum_{j=1}^{n} \lambda_j y_{tj} \geq \delta_t y_t \qquad t = 1,\ldots,s; \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \delta_t y_{ro} \qquad t = 1,\ldots,s \quad ,r = 1,\ldots,s \quad ,r \neq t; \\
& \sum_{j=1}^{n} \lambda_j = 1; \\
& \sum_{t=1}^{s} \delta_t = 1; \\
& \delta_t \in \{0,1\} \qquad t = 1,\ldots,s \\
& \lambda_j \geq 0 \qquad j = 1,\ldots,n.
\end{aligned} \tag{3}$$

In which h is small increment for the kth input DMU_o. Since only one δ_t is equal "1" (Since only for an index l, δ_l is equal "1" and the other $\delta_t (t \neq l)$ are zero), therefore just one output for each unit under evaluation is increased (decreased). In the other words to maintain unit profitability, δ_t determines which y_t should change. In model (3), Due to the changes the objective function shows the maximum profitability. Note that model (3) is a non-linear programming. This model can be transformed into linear programing by $\delta_t y_t = y_t^*$.

$$
\begin{aligned}
Z = Max \ & \frac{\sum_{t=1}^{s} p_t y_t^* + \sum_{t=1}^{s} \delta_t \sum_{\substack{r=1 \\ r \neq t}}^{s} p_r y_{ro}}{\sum_{\substack{i=1 \\ i \neq k}}^{m} c_i x_{ij} + (x_{ko} + h)} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{kj} = x_{ko} + h \quad ; \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \quad , i \neq k \\
& \sum_{j=1}^{n} \lambda_j y_{tj} \geq y_t^* \quad t = 1, \dots, s; \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \delta_t y_{ro} \quad t = 1, \dots, s \quad , r = 1, \dots, s \quad , r \neq t; \\
& \sum_{j=1}^{n} \lambda_j = 1; \\
& \sum_{t=1}^{s} \delta_t = 1; \\
& \delta_t \in \{0,1\} \quad t = 1, \dots, s \\
& \lambda_j \geq 0 \quad j = 1, \dots, n.
\end{aligned} \tag{4}
$$

when $\delta_t = 1$ We expect $y_t = y_t^*$. Additionally, considering binary δ_t, it is clear $\delta_r = 0 (r \neq t)$. To establish the stated condition, the constraints in equation (5) are applied to the model.

$$
\begin{aligned}
& y_t^* = y_t - M(1 - \delta_t) \\
& y_t^* \leq y_t \leq y_t^* + M(1 - \delta_t) \\
& 0 \leq y_t^* \leq M \delta_t
\end{aligned} \tag{5}
$$

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$$Z = Max \frac{\sum_{t=1}^{s} p_t y_t^* + \sum_{t=1}^{s} \delta_t \sum_{\substack{r=1 \\ r\neq t}}^{s} p_r y_{ro}}{\sum_{\substack{i=1 \\ i\neq k}}^{m} c_i x_{ij} + (x_{ko}+h)}$$

$$\begin{aligned}
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{kj} = x_{ko} + h \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} && i=1,\dots,m \quad , \ i \neq k \\
& \sum_{j=1}^{n} \lambda_j y_{tj} \geq y_t - M(1-\delta_t) && t=1,\dots,s \qquad (6) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \delta_t y_{ro} && t=1,\dots,s \quad , r=1,\dots,s \quad , r\neq t \\
& \sum_{j=1}^{n} \lambda_j = 1; \\
& \sum_{t=1}^{s} \delta_t = 1; \\
& y_t^* = y_t - M(1-\delta_t) \\
& y_t^* \leq y_t \leq y_t^* + M(1-\delta_t) \\
& 0 \leq y_t^* \leq M\delta_t \\
& \delta_t \in \{0,1\} && t=1,\dots,s \\
& \lambda_j \geq 0 && j=1,\dots,n.
\end{aligned}$$

By solving the LP in equation (6), the optimal value of y_t is determined.

Theorem1. The new input/output DMU_o $(x_{1o},\dots,x_{ko}+h,\dots,x_{mo},y_{1o},\dots y_t^*,\dots,y_{so})$ is a point on the frontier.

Proof. Suppose in contrary that $z^* > 1$. Since the z^* is the optimal value of (4) thus

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$$\begin{cases} \sum_{j=1}^{n} \lambda_j x_{kj} = \bar{x}_{ko} + h \\ \sum_{j=1}^{n} \lambda_j x_{ij} = \bar{x}_{io} \\ \sum_{j=1}^{n} \lambda_j y_{tj} = \bar{y}_t^* \\ \sum_{j=1}^{n} \lambda_j y_{rj} = \delta_t \bar{y}_{ro} \end{cases}$$

is a feasible solution for model (4). The goal is to maximize the objective function with respect to the small change in a particular input, it should be possible to get the maximum amount possible for the desired output ($\bar{y}_t$). Clearly $\bar{y}_t > y_t^*$ and it is optimal to (4). With this contradiction, the proof is complete.

4 An Application

Banking efficiency is widely recognized as a key determinant of economic performance. Numerous economists argue that inefficiencies or insolvencies in the banking sector can have widespread adverse effects, potentially destabilizing entire economies. Consequently, the efficient operation of banks plays a critical role in ensuring a country's financial stability and economic resilience.

In this section, we demonstrate the practical implementation of the Mixed Marginal Rate (MMR) model using real-world data from the Iranian banking sector. Specifically, we analyze the operational data of 78 branches of a major Iranian bank over the course of one month.

Based on the data provided by the bank, each branch in the dataset utilizes two inputs:

- **Personnel Costs** (x_1: representing labor-related expenditures)
- **Resources** (x_2 : referring to total deposit or funding capacity)

These inputs are used to generate two outputs**:**

- **Expenses** (y_1: operating expenditures incurred by the branch)
- **Income** (y_2: revenues generated from banking activities)

The application of the MMR model to this dataset allows us to:

- Identify which outputs are most sensitive to marginal changes in inputs,
- Determine the optimal output level required to maintain profitability and technical efficiency under minor resource fluctuations,
- Compare efficient branches and uncover patterns that can guide managerial improvements.

In the subsequent sections, we present the computational results, interpret the model's findings, and discuss their implications for decision-making within the banking sector.

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Table 1 The dataset sourced from 78 branches of the Central Bank of Iran

NO.	X1	X2	Y1	Y2	EFF	NO.	X1	X2	Y1	Y2	EFF
1	9285	3375890	2124573	60719	0.2662	40	4356	1396035	780820	21114	0.2361
2	6812	2433266	775666	19468	0.1344	41	4147	934410	333998	7556	0.1652
3	5012	1859182	918456	24533	0.2086	42	7272	1533051	615822	15476	0.1690
4	5649	3253297	1001575	26694	0.1301	43	4966	1152030	439090	11986	0.1608
5	6390	2154490	723034	19188	0.1416	44	5125	1348726	272846	8145	0.0856
6	5259	1933142	570205	14010	0.1243	45	3330	959299	427330	12939	0.2979
7	7151	2178076	730643	20779	0.1418	46	3080	933265	399285	11388	0.4975
8	3095	986050	547837	13410	0.6109	47	3553	886668	276698	7360	0.1638
9	6816	1924052	470813	13079	0.1033	48	5577	1885668	468511	11629	0.1047
10	4413	1744457	473969	10930	0.1143	49	6925	1458627	957769	27733	0.2773
11	5642	1775200	901623	24557	0.2145	50	3715	901951	325387	7985	0.1730
12	6043	1579970	836463	22875	0.2234	51	4925	1458210	278693	7578	0.0807
13	5083	2029564	412238	11800	0.0859	52	3554	676696	275123	6877	0.2951
14	3177	1163828	191457	5769	0.1449	53	3242	1350694	291597	8254	0.166
15	4572	806052	194774	6114	0.1326	54	4286	1057805	628112	21264	0.2521
16	3826	677211	442280	10976	0.4731	55	2919	901177	298605	6482	1.0000
17	4930	1176557	388432	9868	0.1390	56	4343	1131913	279634	7130	0.1041
18	3600	984504	268114	6354	0.1375	57	4953	1242300	419937	10539	0.1423
19	6440	1296710	582007	16374	0.1894	58	3636	549132	260907	6483	0.6595
20	5016	1433301	487136	14556	0.1438	59	4965	1096428	714525	20587	0.2752
21	4034	713392	327821	8183	0.3015	60	5399	1874562	779311	21751	0.1757
22	3116	498527	182661	4200	0.1536	61	3505	716740	419862	11697	0.3826
23	5597	1411256	510555	13905	0.1526	62	4760	1811408	1112192	22751	0.2577
24	5692	1300249	627122	17802	0.2036	63	3811	1015151	285369	7500	0.1249
25	4325	1532620	511506	16258	0.1416	64	4888	1164120	317922	7289	0.1147
26	17388	6876745	4385826	212917	1.0000	65	4803	1151745	246359	6458	0.0901
27	3560	952429	318528	7037	0.1603	66	3430	920893	949924	28520	0.6099
28	4946	1150919	653292	17184	0.2392	67	3606	1018612	314394	7532	0.1552
29	5733	1307423	629419	16682	0.2029	68	8187	3301739	2242190	73552	0.2885
30	2930	521121	168837	4341	1.0000	69	4860	1011451	295253	8539	0.1265
31	3109	886277	291861	8074	0.3470	70	3590	854193	363804	10023	0.2205
32	3766	894835	490638	13797	0.2660	71	4335	716133	294956	9030	0.2698
33	3709	1096409	404305	10732	0.1735	72	4735	1132854	515030	14677	0.1920
34	4091	1327620	435154	12600	0.1386	73	4668	1436979	558628	12434	0.1633
35	3698	912173	382097	10710	0.2007	74	4632	1939890	642143	18848	0.1402
36	4013	656133	218962	5961	0.2593	75	3415	1004094	575541	16044	0.3439
37	4383	773806	403677	11963	0.3022	76	3711	1073523	400537	12305	0.1760
38	3188	728457	196209	5480	0.2304	77	3921	1045090	2457606	86262	1.0000
39	5415	1421501	557347	12629	0.1647	78	3781	1139848	760061	14713	0.2990

To begin our empirical analysis, we calculate the profitability of the bank branches using Model (2). Due to confidentiality restrictions, the actual price and cost coefficients are not publicly accessible. Therefore, to ensure model operability without compromising its structural validity, we assume all price and cost coefficients to be equal to 1. This assumption standardizes the analysis and does not affect the relative efficiency or profitability outcomes.

Based on this approach, Columns 6 and 12 of Table 1 identify four branches 26, 30, 55, and 77 as *profit-efficient*.

Next, we apply Model (6) to assess the impact of marginal changes in throughputs (inputs or outputs) on other throughputs and their influence on efficiency, as discussed in Section 2. Specifically, we evaluate the effect of marginal variations in the first input ($h_1 = 100$) and second input ($h_2 = 10000$). The results are summarized in Table 2.

In Table 2, the values denoted by y_1^{N+}, y_2^{N+}, y_1^{N-} and y_2^{N-} represent the new output levels corresponding to a marginal increase, decrease, respectively, in the input variable under

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consideration. The final column indicates which output variable is most influenced by the marginal input change.

From the results in Table 2, we observe that the reduction in the first input ($h_1 = 100$) is *infeasible* for some branches. However, when increased, the first output of Branch 55 rises significantly from 298,605 to 508,298.45, demonstrating a considerable positive marginal impact.

$$\text{MR}(Branch\ 55) = \frac{\partial \text{y}_1^{N+}}{\partial x} = \frac{\text{y}_1^{N+} - y}{h} = \frac{508298.45 - 298605}{100} = 2096.9345$$

Similarly, as shown in Table 3, a reduction in the second input ($h_2 = 10000$)) is also infeasible. However, increasing leads to a rise in the first output of Branch 30, from 168,837 to 172,858.4.

$$\text{MR}(Branch\ 30) = \frac{\partial \text{y}_1^{N+}}{\partial x} = \frac{\text{y}_1^{N+} - y}{h} = \frac{172858.4 - 168837}{10000} = 0.40214$$

Given the positive value of the gradient, it can be concluded that an increase or decrease in one input variable leads to a corresponding increase or decrease in another variable.

These findings illustrate how marginal input adjustments can impact output levels and provide insight into the trade-offs and sensitivity within efficient production units.

Table 2 Increase and decrease of Personnel Costs

NO	$x_1 + h_1$	$x_1 - h_1$	y_1^{N+}	y_2^{N+}	y_1^{N-}	y_2^{N-}
26	17488	17288	4385826	212917	4385826	212917
30	3030	2830	168837	4341	168837	4341
55	3019	2819	508298.5	6482	298605	6482
77	4021	3821	2457606	86262	2457606	86262

Table 3 Increase and decrease of Resources

NO	$x_2 + h_2$	$x_2 - h_2$	y_1^{N+}	y_2^{N+}	y_1^{N-}	y_2^{N-}
26	6886745	6866745	4385826	212917	4385826	212917
30	531121	511121	172858.4	4341	168837	4341
55	911177	891177	298605	6482	298605	6482
77	1055090	1035090	2457606	86262	2457606	86262

5 Conclusion

This study introduces a linear programming model designed to estimate marginal rate (MR) values on the production frontier within the Data Envelopment Analysis (DEA) framework. The proposed methodology enables the identification of output variables most responsive to small input or output variations, captured by a marginal change parameter. By optimizing an objective function with respect to δ_l, the model determines the optimal output adjustment required to preserve feasibility and efficiency. A key advantage of this model lies in its integrated structure, which facilitates the simultaneous evaluation of marginal changes across multiple throughputs. This approach provides a nuanced understanding of trade-offs and interdependencies within production systems, offering a powerful decision-support tool for performance improvement.

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Furthermore, the model is highly versatile, accommodating a wide range of technological settings and enabling the estimation of various types of marginal rates including those aligned with profitability frontiers. Its ability to classify throughputs into different categories supports strategic decision-making by identifying areas with the highest potential for marginal improvement. The applicability and practical value of the proposed methodology were demonstrated through an empirical case study involving 78 branches of an Iranian bank. The analysis showcased how marginal adjustments in inputs could significantly impact output performance, particularly for profit-efficient units, thereby highlighting the model's relevance in real-world decision environments.

In conclusion, the proposed Mixed Marginal Rate (MMR) approach contributes both theoretically and practically to the literature on DEA, offering a robust framework for marginal analysis that aligns well with managerial priorities in complex production systems.

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