International Journal of Applied Operational Research
Vol. 4, No. 2, pp. 83-88, Spring 2014

Journal homepage: www.ijorlu.ir

Quadratic bi-level programming problems: a fuzzy goal programming approach

M. Saraj, S. Sadeghi*

Received: 11 September 2013 ; **Accepted:** 14 January 2014

Abstract This paper presents a fuzzy goal programming (FGP) methodology for solving bi-level quadratic programming (BLQP) problems. In the FGP model formulation, firstly the objectives are transformed into fuzzy goals (membership functions) by means of assigning an aspiration level to each of them, and suitable membership function is defined for each objectives, and also the membership functions for vector of fuzzy goals of the decision variables controlled by decision maker at the first level are developed in the model formulation of the problem. To achieve the highest membership value of each of the fuzzy goals, we formulate the problem by minimizing the negative deviational variables and thereby obtaining the most satisfactory solution for all decision makers. A numerical example is given to demonstrate the proposed approach.

Keywords Bi-Level Programming, Quadratic Programming, Goal Programming, Fuzzy Goal Programming.

1 Introduction

The bi-level programming problems (BLP) are hierarchical optimization problems in the sense that their constraints are defined in part by a second parametric optimization problem. We can also define BLP as a sequence of two optimization problems in which the feasible region of the upper level problem is determined implicitly by the solution set of the lower level problem. The decision maker at the upper level is termed the leader, and at the lower level, the follower. The leader and the follower each try to optimize their own objective function, but the decision affects the objective value at the other level [1].

A bi-level decision problem for $x \in X \subseteq R^n$, $y \in Y \subseteq R^m$ is formulated as follows [2]:

$$\begin{aligned} &\underset{x \in X}{Min}\, f_1(x, y) \\ &s.t. \\ &\qquad G(x, y) \le 0, \\ &\underset{y \in Y}{Min}\, f_2(x, y) \\ &s.t. \\ &\qquad g(x, y) \le 0, \end{aligned} \tag{1}$$

* **Corresponding Author.**
E-mail: s.sadeghi336@ymail.com (S. Sadeghi)

M. Saraj
Associate Professor, Faculty of Mathematical Sciences and Computer, Shahid Chamran University, Ahvaz, Iran.

S. Sadeghi
M.Sc, Faculty of Mathematical Sciences and Computer, Shahid Chamran University, Ahvaz, Iran.

where

$f_i(x,y) = c_i x + d_i y + (x,y)Q_i(x,y)^t,\ i = 1,2.$

Where Q_1 and Q_2, are constant symmetric matrices, f_1 and f_2 are the upper level (leader) and the lower level (follower) objective functions, respectively, while the vector-valued functions $G: R^n \times R^m \to R^p$ and $g: R^n \times R^m \to R^p$ are called the upper level and lower level constraints, respectively. We also denote the set of the constraint set of the problem by Ω.

Goal programming (GP) was originally proposed by Charnes and Cooper [3]. Fuzzy goal programming (FGP) involves applying the fuzzy set theory to goal programming, thus allowing the model to take into account the vague aspirations of a decision-maker. FGP approach studied by Mohamed [4] is an important technique in dealing with conflicting objectives of decision makers for satisfying decisions for overall benefit of the organization. In the present study, the FGP method is used to solve BLQP problem.

In this paper we investigate the solution of a quadratic bi-level programming problem by fuzzy goal programming method. The organization of the paper is as follows; following the introduction, this paper presents the formulation of QBLP in a FGP framework in Section 2. To illustrate the proposed methodology, a numerical example is considered in Section 3 and Section 4 deals with concluding remarks.

2 **Fuzzy goal programming formulation**

Let $(x^1, y^1; f_1^l)$ and $(x^2, y^2; f_2^l)$ be the optimal solutions of the leader and follower, respectively, when calculated in isolation over the feasible solution space Ω.

The solutions usually are different because of conflicts of nature between two objectives. Therefore, it can easily be assumed that all values larger than or equal to $f_1^u (= f_1(x^2, y^2))$ and all values larger than or equal to $f_2^u (= f_2(x^1, y^1))$ be absolutely unacceptable to leader and follower, respectively.

Let $d_i = | f_i(x,y) - f_i^l |,\ i = 1,2$ be the distance function with unit weight. This distance depends upon (x,y). At $(x,y) = (x^i, y^i)$ (optimal point in Ω-space), $d_i = 0$ and as $f_i(x,y) = f_i^u$, we get the maximum value of d_i as:

$$\overline{d_i} = | f_i^u - f_i^l |,\quad i = 1,2,$$

and

$$\overline{p} = \sup\{\overline{d_i}\},\quad i = 1,2.$$

Thus the membership function for the fuzzy goals can be defined as [5], [6]:

$$\mu_i(d_i(x,y)) = \begin{cases} 0 & d_i(x,y) \geq 0 \\ \dfrac{p - d_i(x,y)}{p} & 0 < d_i(x,y) < p \\ 1 & d_i(x,y) \leq 0 \end{cases} \tag{2}$$

2.1 GP formulation

In a decision making situation, the aim of each DM is to achieve the highest membership value (unity) of the associated fuzzy goal in order to obtain the absolute satisfactory solution. However, in real practice, achievement of all membership values to the highest degree (unity) is not possible due to conflicting objectives. Therefore, decision policy for minimizing the regrets of the DMs for all the levels should be taken into consideration. Therefore, each DM should try to maximize his or her membership function by making them as close as possible to unity by minimizing its negative-deviational variables.

In a fuzzy programming (FP) approach, the highest degree of membership function is one. So, as in Mohamed [4], for the defined membership functions in the flexible membership goals for both the levels can be presented as

$$\frac{p-d_i(x,y)}{p}+d_i^- - d_i^+ = 1.\ \ i=1,2 \tag{3}$$

Where $d_i^-(\geq 0)$ and $d_i^+(\geq 0)$ represent the under and over deviational variables, respectively, from the aspired levels. Now it can be easily realized that the membership goals in expression (3) are inherently nonlinear equation, and this may reduce computational difficulties in the solution process. The membership goals with aspired level 1 can be presented as:

$$-f_i(x,y)+f_i^l+pd_i^- - pd_i^+ = 0,\ \ i=1,2. \tag{4}$$

Now, to build the membership functions for the fuzzy goals of the decision variables controlled by leader, the optimal solution x^{1*} of the upper level problem should be determined first.

The purpose of GP is to minimize the deviations between the achievement of goals and their aspiration levels. Now, we can get the solution of the first level problem by solving the following problem:

$$\begin{aligned}
&Find\ (x,y)\ so\ as\ to\\
&Min\ \ Z = d_1^-\\
&s.t.\\
&\quad -f_1(x,y)+f_1^l+pd_1^- - pd_1^+ = 0,\\
&\quad G(x,y)\leq 0,\\
&\quad g(x,y)\leq 0.
\end{aligned} \tag{5}$$

the above model can be easily solved by the nonlinear techniques, the optimal solution is assumed to be $(x^{1*},y^{1*};f_1^*)$ In this way, the range of decision variable x should be around x^{1*} with maximum tolerance $t=(t^L,t^R)$, and the following membership function specify x, as [7]:

$$\mu_i(\mu_{x^*}(x))=\begin{cases}\dfrac{x-(x^{1*}-t^L)}{t^L} & x^{1*}-t^L<x<x^{1*}\\[2ex] \dfrac{x^{1*}+t^R-x}{t^R} & x^{1*}<x<x^{1*}+t^R\end{cases} \tag{6}$$

Therefore, the membership goal with the aspired level one can be expressed as:

$$\begin{aligned} &\frac{x-(x^{1*}-t^L)}{t^L}+d_L^- - d_L^+ =1, \\ &\frac{x^{1*}+t^R-x}{t^R}+d_R^- - d_R^+ =1, \end{aligned} \tag{7}$$

where $d_L^-, d_R^-, d_L^+, d_R^+ \geq 0$ with $d_L^- \times d_L^+ = 0, d_R^- \times d_R^+ = 0$, represent the under and over deviational, respectively, from the aspired levels.

Now, considering the goal achievement problem of the goals at the same priority level, an equivalent fuzzy bi-level quadratic goal programming model of the problem can be proposed as:

$$\begin{aligned} &Find\ (x,y)\ so\ as\ to \\ &Min \quad ZZ = w_{f_1} d_1^- + w_{f_2} d_2^- + w_L (d_L^- + d_L^+) + w_R (d_R^- + d_R^+) \\ &s.t. \\ &\quad -f_i(x,y)+f_i^{\,l}+pd_i^- - pd_i^+ = 0, \quad i=1,2. \\ &\quad \frac{x-(x^{1*}-t^L)}{t^L}+d_L^- - d_L^+ =1, \\ &\quad \frac{x^{1*}+t^R-x}{t^R}+d_R^- - d_R^+ =1, \\ &\quad G(x,y) \leq 0, \\ &\quad g(x,y) \leq 0. \end{aligned} \tag{8}$$

where ZZ represents the fuzzy achievement function consisting of the weighted under deviational variables d_i^- $i=1,2$ and the under and over deviational variables d_L^-, d_L^+, d_R^- and d_R^+ for the fuzzy goals of all the ULDM variables, where the numerical weights $w_{f_i}\ (i=1,2)$, w_L and w_R represent the relative importance of achieving the aspired levels of the respective fuzzy goals subject to the constraints set in the decision situation.

In the present formulation, numerical weights $w_{f_i}\ (i=1,2)$, w_L and w_R are determined as [4]:

$$w_{f_i} = \frac{1}{f_i^u - f_i^l},\ i=1,2, \quad w_L = \frac{1}{t_L}, \quad w_R = \frac{1}{t_R}$$

4 Numerical example

Consider the following QBLP problem [8]:

$$\begin{aligned} &\underset{x \in X}{Min}\, f_1(x,y) = x_1^2 - 3x_2 - 4y_1 + y_2^2 \\ &s.t. \quad x_1^2 + 2x_2 \leq 4, \\ &\underset{y \in Y}{Min}\, f_2(x,y) = 2x_1^2 + y_1^2 - 5y_2 \\ &s.t. \quad x_1^2 - 2x_1 + x_2^2 - 2y_1 + y_2 \geq -3, \\ &\qquad x_2 + 3y_1 - 4y_2 \geq 4. \end{aligned}$$

The individual optimal solutions of the leader and follower are $(x_1^1, x_2^1, y_1^1, y_2^1) = (0, 2, 4, 1)$ with $f_1^L = -21$ and $(x_1^2, x_2^2, y_1^2, y_2^2) = (0, 2, 1.87, 0.9)$ with $f_1^L = -1$, respectively. The lower tolerance limits of the goals are defined as $f_1^u = -12.67$ and $f_2^u = 11$.

The optimal solution of upper level problem based on (5) is $(x^{1*}, x^{2*}, y^{1*}, y^{2*}) = (0, 2, 3.99, 0.99)$ Let the upper level DM decide $(x^{1*}, x^{2*}) = (0, 2)$ with positive tolerance $d_1^{R+} = 1, d_2^{R+} = 1$ and weight of $w_1^R = w_2^R = \frac{1}{1} = 1$, (one-sided membership function).

The satisfactory solution of the DBL-MOLP problem based on (8) is $(x^*, y^*) = (0, 2, 2.9, 1.6)$ with objective functions values $f_1 = -15$ and $f_2 = 0.41$ with membership functions values $\mu_{f_1} = 0.5$ and $\mu_{f_1} = 0.89$, respectively.

4 Discussion and Conclusion

Various proposed methods for solving the bilevel programming problems can be classified into the following five categories [9]: extreme-point search; transformation approach; descent and heuristic; intelligent computation and interior point. Jie Lu et al. [10] point out that there are two fundamental issues in both bilevel decision theory and practice. One is how to model a real-world bilevel decision problem that may have various situations at the two decision levels, and the other is how to find an satisfactory solution for the decision problem for the bilevel programming problems, which are generally difficult to solve due to the complexity of the problem. It has been proved that solving the bilevel linear programming is an NP-hard problem [2,8] and even it is an NP-hard problem to search for the locally optimal solution of the bilevel linear programming [8]. Thus, it is difficult to solve the bilevel programming, especially to solve the bilevel quadratic programming. For the above reasons, the most existing numerical techniques are effective only for bilevel quadratic programming with special structure or obtaining the local solution for bilevel nonlinear programming.
Presently many approaches have been made to solve BLQP mostly based on KKT programming.

The fundamental deficiency of current Kuhn–Tucker approach remains is that it could not well solve a quadratic BLP problem when constraint functions at the upper-level are of arbitrary quadratic form [2].

In [8], Guangmin Wang and et al., proposes an approximate programming algorithm to solve bilevel nonlinear programming problem. In algorithm, the perturbed Fischer-Burmeister function is adopted, which can avoid the difficulty of dealing with the non-differentiable because of the complementary condition. And the simplex method is adopted to solve the approximate problem to the nonlinear programming transformed from the bilevel nonlinear programming.

However, it has not yet made a significant breakthrough in the field of BLQPP. In this paper we apply FGP method to solve BLQPP. Following are the advantages and special features of the proposed FGP approach to BLQPP:

1. In the well-known GP model, it is often difficult for the DM to decide the desired levels for the goals. Our proposed FGP approach presents an easy way to determine suitable aspiration levels instead of relying solely on the DMs experience which may lead to an inaccurate solution and increase the solution time.
2. In practical applications, the required data of BLQPP may be imprecise. Thus, adaptation of fuzzy sets theory in the solution process increases the flexibility and effectiveness of the FGP approach.
3. Using GP in conjunction with fuzzy set theory for solving BLQPP provides a satisfactory solution. Adding interactive programming to this combination leads to the preferred satisfactory solution. Thus, model (8) can be considered as an integrated and easy mathematical model to solve BLQPP problems.

4. Most advantageous feature of our FGP approach is that one can easily apply it to non-linear programming problems. This method is very effective at finding satisfactory solution or near optimal solution.

In this paper, a FGP approach is proposed to solve BLQP problems. This technique can be easily extended for other BLQP problems where the decision variables are integers. In the same fashion the present problem can be also considered and extended for the case of bi-level quadratic fractional programming problems.

References

1. Miller, T., Friesz, T., Tobin, R., (1992). Heuristic algorithms for delivered price spatially competitive network facility location problems. Ann .Oper .Res, 34, 177–202.
2. Bard, J. F., (1991). Technical note: some properties of the bi-level programming problem. Journal of Optimization Theory and Application, 68, 371–378.
3. Charnes, A., Cooper W. W., (1961). Management model and industrial application programming. Wiley, New York, 1.
4. Mohamed R. H., (1997). The relationship between goal programming and fuzzy programming. Fuzzy Sets and Systems, 89, 215–222.
5. Gupta, S., Chakraborty M., (1997). Multiobjective linear programming: a fuzzy programming approach. Int J Manag Syst, 13(2), 207–214.
6. Kailash, L., (2012). Fuzzy Goal Programming Approach to Multi Objective Quadratic Programming Problem. Proc Nat Acad Sci India Sect A, 82(4), 317-322.
7. Osman, M. S., Abo-Sinna M. A., Amer, A. H., Emam, O. E., (2004). A multi-level nonlinear multi-objective decision making under fuzziness. Journal of Applied Mathematics and Computation, 153, 239–252.
8. Guangmin, W., Kejun Zhua., Zhongping, W., (2010). An approximate programming method based on the simplex method for bi-level programming problem. Computers and Mathematics with Applications, 59, 3355-3360.
9. Shih, H. S., Wen, U. P., Lee, E. S., et al., (2004). A neural network approach to multiobjective and multilevel programming problems. Computers and Mathematics with Applications, 48, 95-108.
10. Jie Lu., Chenggen Shi., Guangquan Zhang., (2006). On bilevel multi-follower decision making: general framework and solutions, Information Sciences, 176, 1607-1627.