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Nonparametric Shewhart-type quality control charts in fuzzy environment

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Abstract Nonparametric control charts are presented in order to figure out the problem of detecting changes in the process median (or mean), or changes in the variability process where there is limited knowledge regarding the underlying process. When observations are reported imprecise, then it is impossible to use classical nonparametric control charts. This paper is devoted to the problem of constructing nonparametric control charts in presence of fuzzy data and parameters. For this aim, we propose two different methods: distance between fuzzy numbers and credibility measure for ranking fuzzy data, that can be used to construct sign and Wilcoxon signed-rank control charts. Then, this statistical control charts are applied to monitor location and scale parameters of a continuous statistical process. Finally, proposed control charts application is evaluated with numerical examples.

Keyword: Fuzzy Sign Control Chart, Synthetic Control Chart, Average Run Length, $D_{p,q}-$Distance, Credibility Measure.

1 Introduction

A control chart is one of the most used tools for monitoring the central and dispersion parameters in a manufacturing process. The most common type of these statistic control charts is Shewhart- type control chart. In the Shewhart- type control charts, the control limits is derived based upon parametric assumptions as normal distribution.

Hence, in the functional issues due to lack enough assumptions about the statistical process distribution, using traditional control charts in these situations can have negative effects.

Nonparametric control charts are parallel alternatives if one is concerned about the non- normality process. A nonparametric control chart is defined in term of run length distribution when the process is in- control.

Amin et al. [1] considered nonparametric charts for process median based on the sign test. They compared the parametric shewhart chart and sign (SN) chart for various shift sizes and underlying distributions based on their average run length (ARL). The main practical

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advantage of the SN chart is that the false alarm rate (FAR) and average run length (ARL_0) remain the same for all continuous distributions, this is not true for shewhart chart.

Bakir [2] presented a shewhart type nonparametric control chart for monitoring the median of a continuous symmetric distribution using Wilcoxon signed-rank (SR) statistic. He showed that the SR chart can compete well with the parametric shewhart chart. In fact, for some heavy-tailed symmetric distributions, the SR chart is more efficient than parametric chart.

Amin et al. [1] provided nonparametric control chart for detecting changes in the variability process based on sign test .Khilare and Shirke [12] proposed synthetic control chart which combines sign chart presented by Amin et al. [1] and conforming run length (CRL) chart suggested by Bourke [3].

Precise data are not always available. After the introduction of fuzzy sets theory by Zadeh [27], Wang and Raz [25] and Raz and Wang [20] proposed two approaches for the construction of control charts, namely a probabilistic approach and a membership approach. In the probabilistic approach, first the representative values of fuzzy observations computed and then utilized to construct control charts using traditional statistical methods. Kanagawa et al. [10] proposed linguistic control charts to monitor the process average and variability. They aimed directly at controlling the underlying probability distributions of linguistic data. $\alpha-$cut fuzzy control charts for attributes were developed by Gulbay and Kahraman [9]. $\alpha-$cut approach provides the ability to determine the tightness of the inspection by selecting a suitable level. Gulbay and Kahraman [8] developed a direct fuzzy approach (DFA) to fuzzy control charts without any defuzzification and then defined fuzzy unnatural pattern rules based on the probabilities of fuzzy events. Also, they [7] proposed a method to construct fuzzy control charts for categorical data in which the linguistic data are changed into representative values. They used the three-sigma rule to compute fuzzy control limits having a strong base founded upon the properties of the normal distribution, i.e., on the three-sigma rule. Nguyen et al. [18] proposed a detailed procedure to classify a process, but some of their rules were found indistinguishable by Nguyen et al. [19], who later proposed a remedy for a better performance.

Faraz and Moghaddam [4] presented a fuzzy control chart for controlling the process mean with a warning line. The warning line designed for detecting desired shift in the process mean that is important to the company. Also, Faraz et al. [5] introduced a control chart for monitoring variables and showed that the control limits in classical Shewhart charts must be adjusted when there is on ambiguity in the process mean beside randomness. Senturk and Erginel [22] propose fuzzy $\bar{X}$ -R and $\bar{X}$ -S control charts with $\alpha-$cuts. Their idea can be traced to traditional $\bar{X}$ -R and $\bar{X}$ -S control charts, which rely on the properties of the normal distribution. Unfortunately, this assumption is not addressed in their paper. It should be noted that, in the presence of fuzzy data, the variance of normal observations is increased (see, Kurner [13]). Shu and Wu [24] proposed fuzzy $\bar{X}$ -R control charts whose fuzzy control limits are obtained based on the result of the resolution identity. They utilized fuzzy dominance approach, which directly compares the fuzzy sample mean to the fuzzy control limits to determine the process condition. Recently, Zabihinpour et al. [26] developed a fuzzy mean and range control charts. In their approach, the observations and control limits are triangular fuzzy numbers. Instead of using transformation or defuzzifications techniques to determine the process condition, they proposed a direct approach based on the percentage of area of the sample mean which remains outside the control limits. The non-parametric control charts are based upon accurate data and conditions. Although in uncertain conditions, these charts are mostly used in quality control, no research has been done in this regard so far. The goal of this research is to identify non-parametric control charts in a fuzzy environment to monitor the location and scale parameters that not only do not need the assumptions of parametric models and defuzzification functions but also have a satisfactory performance.

This paper is organized as follows. In Section 2, we recall some definitions of fuzzy number, distance between them and credibility measure theory. In Section 3, we present sign and Wilcoxon signed-rank control charts based on distance and credibility measure ranking methods in order to monitor the median process continuously. We also explain their application with numerical examples. In Section 4, we utilize sign control chart for monitoring variability process when process information is reported as imprecise data. We then describe its application with a numerical example. Section 5, presents results and conclusion.

2 Fuzzy Numbers

In this section, few definitions of fuzzy numbers, distance between two fuzzy numbers, credibility measure and ranking of fuzzy variables are explained.

2.1 Fuzzy numbers

A fuzzy subset $\tilde{A}$ of the universal set χ is defined by its membership function $\mu_{\tilde{A}}$: $\chi \to [0,1]$,with the set $\mathrm{supp}(\tilde{A}) = \{x \in \chi : \mu_{\tilde{A}}(x) > 0\}$, the support of $\tilde{A}$. In this work, $\mathbb{R}$ (the real line) is considered as the universal set. It is denoted by $\tilde{A}[\alpha]$ the α -cut of the fuzzy subset $\tilde{A}$ of $\mathbb{R}$, is defined for every $\alpha \in [0,1]$, by $\tilde{A}[\alpha] = \{x \in \mathbb{R}, \mu_{\tilde{A}}(x) \geq \alpha\}$, and $\tilde{A}[0]$ is the closure of $\mathrm{supp}(\tilde{A})$. The fuzzy subset $\tilde{A}$ of $\mathbb{R}$ is called a fuzzy number if for every $\alpha \in [0,1]$, the set $\tilde{A}[\alpha]$ is a non-empty compact interval. Such an interval will be denoted by $\tilde{A}[\alpha] = [\tilde{A}_\alpha^L, \tilde{A}_\alpha^U]$, where $\tilde{A}_\alpha^L = \inf\{x : x \in \tilde{A}_\alpha\}$ and $\tilde{A}_\alpha^U = \sup\{x : x \in \tilde{A}_\alpha\}$. The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.

One of the most popular types of a fuzzy number, being considered in this work, is the recalled trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)_T$ whose membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x < a_2 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x < a_4 \\ 0, & x > a_4 \end{cases}$$

If $a_2 = a_3$, it is called a triangular fuzzy number and denoted by $\tilde{A} = (a_1, a_2, a_3, a_4)_T$. For more detailed information regarding fuzzy numbers, see Lee [15].

2.2 $D_{p,q}$-Distance between two fuzzy numbers

Definition 1 The $D_{p,q}$-distance, indexed by parameters $1 \le p < \infty$ and $0 \le q < 1$, between two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is a nonnegative function given as follows [23]:

$$D_{p,q}\left(\tilde{A},\tilde{B}\right) = \begin{cases} \left[\left(1-q\right)\int_0^1 \left|A_\alpha^- - B_\alpha^+\right|^p d\alpha + q\int_0^1 \left|A_\alpha^+ - B_\alpha^+\right|^p d\alpha\right]^{\frac{1}{p}}, & p < \infty \\ \left(1-q\right) \sup\limits_{0<\alpha\le 1}\left(\left|A_\alpha^- - B_\alpha^-\right|\right) + \left(q\right) \inf\limits_{0<\alpha\le 1}\left|A_\alpha^+ - B_\alpha^+\right|, & p = \infty \end{cases}$$

The analytical properties of $D_{p,q}$ is depend on the first parameter p, while the second parameter q is the weighted one. $\left(\mathcal{F}\left(\mathbb{R}\right), D_{p,q}\right)$ is a complete metric space. If there is no reason for distinguishing any side of the fuzzy numbers, $D_{2,\frac{1}{2}}$ is recommended.

For instance, for trapezoidal fuzzy numbers $\tilde{A} = (a_1, a_2, a_3, a_4)_T$ and $\tilde{B} = (b_1, b_2, b_3, b_4)_T$ the above distance with $p = 2$ and $q = \frac{1}{2}$ is calculated as [19]:

$$D_{2,\frac{1}{2}}\left(\tilde{A},\tilde{B}\right) = \sqrt{\frac{1}{6}\left[\sum_{i=1}^{3}(b_i - a_i)^2 + (b_2 - a_2)^2 + \sum_{i\in\{1,2\}}(b_i - a_i)(b_{i+1} - a_{i+1})\right]} \tag{2.1}$$

Example 1 Let $\tilde{A} = \left(1.5, 2, 2.7\right)_T$ and $\tilde{B} = \left(0.5, 0.9, 1.8\right)_T$ be two fuzzy triangular numbers. The $D_{2,\frac{1}{2}}\left(\tilde{A},\tilde{B}\right)$ is obtained as $D_{2,\frac{1}{2}}\left(\tilde{A},\tilde{B}\right) = \sqrt{\frac{1}{6}(1 + 2.42 + 0.25 + 1.1 + 0.55)} = 0.942.$

2.3 Credibility Measure

Here we introduce an index proposed by Liu [16] which is also the foundation of the very popular credibility theory. For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the map $Cr : \mathcal{F}\left(\mathbb{R}\right) \times \mathcal{F}\left(\mathbb{R}\right) \to [0,1]$ which is defined as follows:

$$Cr\left(\tilde{A} > \tilde{B}\right) = \frac{1}{2}\left[\sup_{x,y:x>y} \min\left\{\mu_{\tilde{A}}\left(x\right), \mu_{\tilde{B}}\left(y\right)\right\} + 1 - \sup_{x,y:x\le y} \min\left\{\mu_{\tilde{A}}\left(x\right), \mu_{\tilde{B}}\left(y\right)\right\}\right]$$

that $\tilde{A}$ is larger than $\tilde{B}$ is called the credibility degree. In addition, the credibility degree of $\tilde{A} \le \tilde{B}$ is given by

$$Cr\left(\tilde{A} > \tilde{B}\right) = \frac{1}{2}\left[\sup_{x,y:x\le y} \min\left\{\mu_{\tilde{A}}\left(x\right), \mu_{\tilde{B}}\left(y\right)\right\} + 1 - \sup_{x,y:x>y} \min\left\{\mu_{\tilde{A}}\left(x\right), \mu_{\tilde{B}}\left(y\right)\right\}\right]$$

It is easy to verify that the credibility index Cr has the following properties:

lemma 1 For two fuzzy number $\tilde{A}$ and $\tilde{B}$, it is easy to verify that

- $Cr\{\tilde{A} > \tilde{B}\} \in [0,1],$
- $Cr\{\tilde{A} > \tilde{B}\} = 1 - Cr\{\tilde{A} \le \tilde{B}\},$
- $Cr\{\tilde{A} > \tilde{B}\} = 1 \mathit{if\ only\ if}\ \tilde{B}_0^U \le \tilde{A}_0^L.$

3 Nonparametric control chart to monitor location parameter in fuzzy environment

Nonparametric control charts are based on crisp (exact / non fuzzy) observations. However, in practice, there are many situations in which it has been found that the available data are imprecise (fuzzy) quantities rather than being precise (crisp) numbers. Many researchers have designed fuzzy control charts to communicate uncertainly due to fuzziness. Using defuzzification approach reduces the vague observation to exact numbers and it also reduces the informational content of the original fuzzy data. In addition, various defuzzification methods may result in different conclusions about the process.

In practice, there are many situations in which process is to be controlled in the presence of the two kinds of uncertainty (unknown underlying process distribution and imprecise information). In such situation, neither fuzzy nor underlying process distribution can monitor the process adequately individually. In this section, we propose a new approaches for construction control charts that can handle both uncertainties. As shewhart control charts follow parametric assumption (normal distribution) and crisp data, when do not have enough information about population distribution or parametric distribution, one possible solution is using nonparametric inferences for location and scale parameters under condition of available crisp data. If observations are reported as imprecise (fuzzy), then it is impossible to use classical nonparametric control charts. Hence, designed control charts that are unable to control lack of enough parametric assumptions and imprecise data simultaneously, will be significant.

To perform nonparametric control charts for monitoring the median of a continuous process based on imprecise observation, we propose two different methods: $D_{p,q}$- distance ranking and credibility measure, that can be used for computing sign and Wilcoxon signed-rank statistics in every sub group of a continuous process, when observations are reported as imprecise.

3.1 Nonparametric control charts based on $D_{p,q}$ - distance ranking method

Fuzzy nonparametric control chart is based on fuzzy nonparametric tests. Each of this fuzzy nonparametric tests have their advantages. For instance, fuzzy sign tests are applicable due to its simplicity and having the less assumptions on quantiles related continuous distribution, while, fuzzy Wilcoxon signed-rank test is used for median related symmetric and continuous distribution. In order to construct fuzzy nonparametric control charts, at first, we obtain control limits based on classical methods and a nominal specified value. Then, in each subgroup sample, we determine nonparametric statistics based on - distance ranking approach. If sample statistics fall between control limits, we can say, it is in-control (IC) process, otherwise, the process is declared out-of-control state.

3.1.1 Sign control chart based on $D_{p,q}$ - distance ranking method

Fuzzy sign control chart is based on the fuzzy sign test. Let $\tilde{X}_{i1},\tilde{X}_{i2},...,\tilde{X}_{in}$ be a random sample (subgroup) of size $n>1$ observed from a continuous process with median M (crisp or imprecise) at sampling instances $i=1,2,...,m$. It is assumed that the IC process median is known or specified to be equal to $\tilde{M}_0$. We further assume that is $\tilde{M}_0$ known and when $\tilde{M} \neq \tilde{M}_0$ the process is out of control (OoC). A control chart is a graphic consisted of values of a plotting statistic and the associated control limits. The steps for fuzzy sign plotting statistic computation are as follows:

1. Set the arbitrary value $\tilde{B}$, where $\tilde{B}$ is a fuzzy value which is less than the smallest observation among sample observation,
2. Determine the distance between $\tilde{M}$ and $\tilde{B}$ i.e. $D_{2,\frac{1}{2}}(\tilde{M},\tilde{B})$,
3. Compute the distance between $\tilde{X}_{ij}$ and $\tilde{B}$ i.e. $D_{2,\frac{1}{2}}(\tilde{X}_{ij},\tilde{B})$, for $i=1,2,...,m$.
4. The fuzzy sign statistic is obtained by the following formula:

$$SN_i = \sum_{j=1}^{n} sign\left(D_{2,\frac{1}{2}}\left(X_{ij}\ , B\right) - \left(D_{2,\frac{1}{2}}\left(M_0\ , B\right)\right)\right), \qquad i=1,2,...,m \tag{4.1}$$

for $i=1,2,...,m$, where sign (.) is the sign function.

In order to find the control limits, we know that the distribution of classical SN_i is known. This can be easily obtained using the relation $SN_i = 2T_i - n$, where T_i is the usual sign test statistic, which is counting the number of sample observations greater than $\tilde{M}$. Also, the IC distribution of T_i is binomial with parameters n and 0.5. It follows that the IC distribution of SN_i is symmetric about 0, hence the upper and lower control limits and the centerline of the two sided nonparametric sign chart are given by $UCL = c, CL = 0, LCL = -c$, where c is some positive integer between 0 and n.

If the plotting statistic SN_i falls on or outside one of the control limits, that is, if $SN_i \leq -c$ or $SN_i \geq c$, the process is declared to be OoC. The charting constant c is obtained for a specified ALR_0, which in standard known case, is equal to the reciprocal of the nominal false alarm rate (FAR) and is denoted by α . Since the IC distribution of SN_i is symmetric about 0, c is obtained as the smallest integer such that $P\left(SN_i \geq c \mid IC\right) \leq \frac{\alpha}{2}$. For example, for $n = 5$, using the binomial tables, we find $P\left(SN_i \geq 5 \mid IC\right) = 0.0312$, which leads to a $FAR = 0.0624$. This is the lowest attainable FAR for $n = 5$.

3.1.2 Wilcoxon signed-rank control chart based on $D_{p,q}$-distance ranking method

Fuzzy Wilcoxon signed-rank control chart is based on the fuzzy Wilcoxon signed-rank test. Let $\tilde{X}_{i1}, \tilde{X}_{i2}, ..., \tilde{X}_{in}$ be a random sample (subgroup) of size n>1 observed from a continuous and symmetric process with median $\tilde{M}$ (crisp or imprecise) at sampling instances $i=1,2,...,m$. It is assumed that the IC process median is known to be equal to $\tilde{M}$. The steps for fuzzy Wilcoxon signed-rank plotting statistics computation being similar to SN statistic can be summarized as follows:

1. Set the arbitrary value $\tilde{B}$ as observations' origin,

2. Determine $D_{2,\frac{1}{2}}(\tilde{M},\tilde{B})$,

3. Determine $D_{2,\frac{1}{2}}\left(X_{ij},\tilde{B}\right)$ and $D_{2,\frac{1}{2}}\left(X_{ij},M_0\right)$ for $i=1,2,...,m$, $j=1,2,...,n$,

4. Ranking $D_{2,\frac{1}{2}}\left(X_{ij},M_0\right)$ for every subgroup. Note that we use midrank method to deal with tied observations,

5.The final fuzzy signed-rank SR statistics is obtained by the following formula:

$$SR_i=\sum_{i=1}^{n} sign\left(D_{2,\frac{1}{2}}\left(\tilde{X}_{ij},\tilde{\mathrm{B}}\right)-D_{2,\frac{1}{2}}\left(\tilde{M}_0,\tilde{\mathrm{B}}\right)\right)r\left(D_{2,\frac{1}{2}}\left(\tilde{X}_{ij},\tilde{M}_0\right)\right),\ i=1,2,...,\mathrm{m} \tag{4.2}$$

for $i=1,2,...,m$, where $r(A)$ is the A rank value in the sample.

In the classical case, SR statistic is linearly related to the more well-known Wilcoxon signed-rank statistic W^+ through the relation $SR_i=2W^+-\frac{n(n+1)}{2}$, where W^+ is the sum of the ranks of absolute values corresponding to the positive deviations. Because W^+ is known to be free-distribution [6[, so are the SR_i and hence the classical SR chart.

The IC distribution of SR_i is symmetric about 0, thus, choosing $UCL=-LCL$ appears to be reasonable and similar to the fuzzy sign chart results in a symmetric control chart. Thus, the control limits and central line of the two sided SR chart are: $UCL=d, CL=0, LCL=-d$, where d is some positive integer between 1 and $\frac{n(n+1)}{2}$. If the plotting statistic SR_i fall on or outside the control limits, the process is declared to be OoC; otherwise, the process is considered to be IC. The constant d is found so that a specified FAR (or an ALR_0) is attained.

Example 2 We illustrate the sign and Wilcoxon signed-rank control charts using a fuzzy data set related to the inside diameters of piston rings manufactured by a foreign process [11]. There are the 25 samples, each of size five that were collected when the process was thought to be IC, Table 1. We assume that the underlying distribution is symmetric with a known median $M=74$ mm. Table 2 shows the SN_i and SR_i statistics.

The sign and Wilcoxon signed- rank control charts are shown in Figures 1 and 2 with control limits at ± 5 and ± 15, respectively. The results from these charts indicate that the medians of all groups fall between control limits, and we can conclude that the process is IC.

Table 1 TFNs for inside piston diameter measurements

Sample Number	X_a					X_b					X_c				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
1	74.029	74.001	74.018	73.991	74.007	74.03	74.002	74.019	73.991	74.008	74.031	74.003	74.02	73.993	74.009
2	73.994	73.991	74	74.01	74.003	73.995	73.992	74.001	74.01	74.004	73.996	73.993	74.002	74.012	74.005
3	73.987	74.023	74.02	74.004	74.001	73.988	74.024	74.021	74.004	74.002	73.989	74.025	74.022	74.006	74.003
4	74.001	73.995	73.992	74.014	74.008	74.002	73.996	73.993	74.014	74.008	74.003	73.997	73.994	74.016	74.01
5	73.991	74.006	74.014	73.988	74.013	73.992	74.007	74.015	73.988	74.014	73.993	74.008	74.016	73.99	74.015
6	74.008	73.993	73.996	73.984	73.994	74.009	73.994	73.997	73.984	73.995	74.01	73.995	73.998	73.986	73.996
7	73.994	74.005	73.993	73.999	74.004	73.995	74.006	73.994	73.999	74.005	73.996	74.007	73.995	74.001	74.006
8	73.984	74.002	73.992	74.014	73.987	73.985	74.003	73.993	74.014	73.988	73.986	74.004	73.994	74.016	73.989
9	74.007	73.994	74.008	74.004	74.003	74.008	73.995	74.009	74.004	74.004	74.009	73.996	74.01	74.006	74.005
10	73.997	73.999	73.989	74.006	73.994	73.998	74	73.99	74.006	73.995	73.999	74.001	73.991	74.008	73.996
11	73.993	73.997	73.993	73.994	73.989	73.994	74.001	73.994	73.994	74.002	73.995	73.999	73.995	73.996	74.003
12	74.003	73.999	74.006	73.999	73.995	74.004	74	74.007	73.999	73.996	74.005	74.001	74.008	74.001	73.997
13	73.982	74.001	73.997	73.996	74.011	73.983	74.002	73.998	73.996	74.012	73.984	74.003	73.999	73.998	74.013
14	74.005	73.966	73.993	73.999	73.983	74.006	73.967	73.994	73.999	73.984	74.007	73.968	73.995	74.001	73.985
15	74.011	74.013	73.997	73.998	74.006	74.012	74.014	73.998	73.998	74.007	74.013	74.015	73.999	74	74.008
16	73.999	73.983	74.004	73.997	73.995	74	73.984	74.005	73.997	73.996	74.001	73.985	74.006	73.999	73.997
17	73.993	74.011	73.985	74.004	74.006	73.994	74.012	73.986	74.004	74.007	73.995	74.013	73.987	74.006	74.008
18	74.005	74.009	74.017	74.002	73.999	74.006	74.01	74.018	74.002	74	74.007	74.011	74.019	74.004	74.001
19	73.983	74.001	74.002	74.004	73.996	73.984	74.002	74.003	74.004	73.997	73.985	74.003	74.004	74.006	73.998
20	73.999	74.009	74.012	74.019	74.002	74	74.01	74.013	74.019	74.003	74.001	74.011	74.014	74.021	74.004
21	73.981	74	74.014	74.004	73.995	73.982	74.001	74.015	74.004	73.996	73.983	74.002	74.016	74.006	73.997
22	74.003	73.998	73.989	74.005	74.008	74.004	73.999	73.99	74.005	74.009	74.005	74	73.991	74.007	74.01
23	74.009	73.988	73.989	74.008	74.013	74.01	73.989	73.99	74.008	74.014	74.011	73.99	73.991	74.01	74.015
24	74.014	74.007	73.992	73.999	74.009	74.015	74.008	73.993	73.999	74.01	74.016	74.009	73.994	74.001	74.011
25	73.981	73.983	73.994	74.016	74.012	73.982	73.984	73.995	74.016	74.013	73.983	73.985	73.996	74.018	74.014

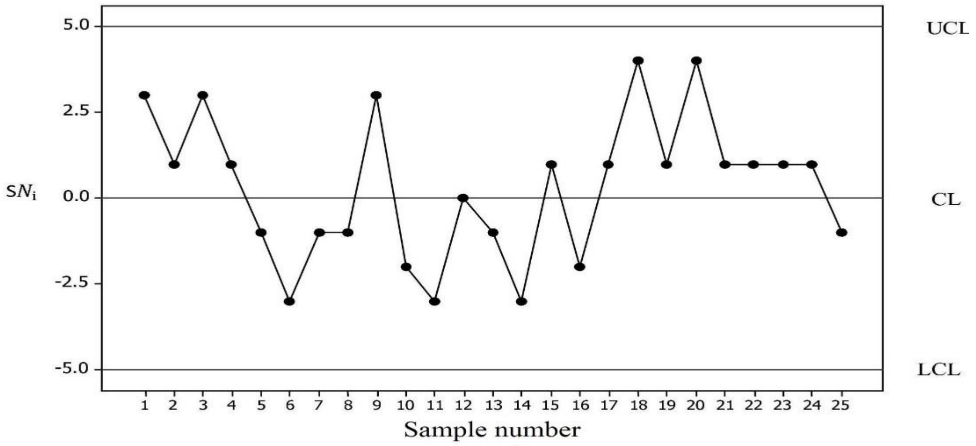

Fig. 1 Shewhart type control chart.

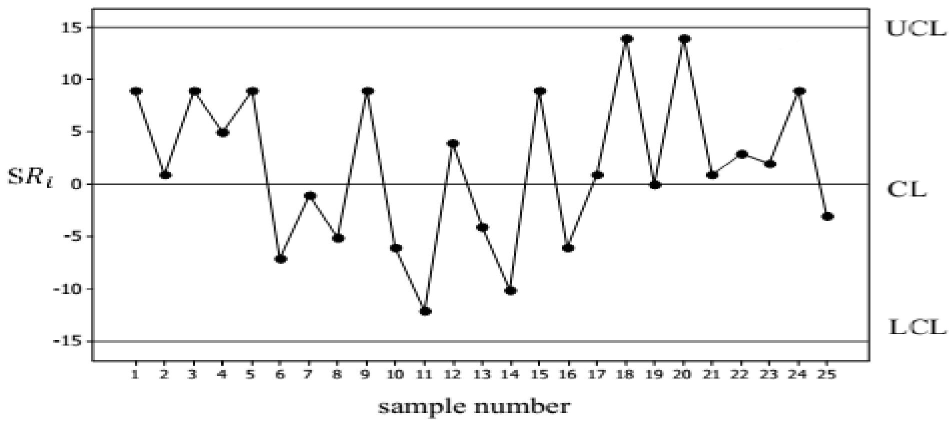

Fig. 2 Shewhart type control chart.

Table 2 SN and SR statistics

Sample number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
SN_i	3	1	3	1	-1	-3	-1	-1	3	-2	-3	0	-1	-3	1	-2	1	4	1	4	1	1	1	1	-1
SR_i	9	1	9	5	9	-7	-1	-5	9	-6	-12	4	-4	-10	9	-6	1	14	0	14	1	3	2	9	-3

3.3 Nonparametric control charts based on credibility measure ranking method

In this subsection, we suggest nonparametric control charts based on credibility measure ranking approach. In this method, nonparametric test statistics are computed as a set, hence we apply concept to make a decision rule due to in control or out of the control statistical process.

P-value is the probability of obtaining a test statistic at least as extreme as the one that was actually observed assuming that the null hypothesis is true. A statistical process can consider as a series of hypothesis tests. Hence, control charts can be thought of as repeated hypothesis tests of the null hypothesis that the process is in control. Considering concept, we propose fuzzy sign control chart and fuzzy Wilcoxon control chart to monitor median of a statistical continuous process.

3.3.1 Fuzzy sign control chart based on credibility measure ranking method

Let $\tilde{X}_{i1},\tilde{X}_{i2},...,\tilde{X}_{in}$ be a random sample (sub-group) of the size $n>1$ observed from a continuous process with median $\tilde{M}$ (crisp or imprecise) at sampling instances $i=1,2,...,m$. It is assumed that the IC process median is known or specified to be equal to $\tilde{M}$. In each sub-group, the fuzzy sign (FSN) plotting statistics is computed as follows:

$$\tilde{t}_i=\{\tilde{t}_i^{\,L},\tilde{t}_i^{\,L}+1,...,\tilde{t}_i^{\,U}\},\quad i=1,2,...,m,$$

in which

$$\tilde{t}_i^{\,L}=\inf_{\alpha>0.5}\sum_{j=1}^{n}I\left(Cr\{\tilde{X}_{ij}\geq\tilde{M}\}\right),\qquad \tilde{t}_i^{\,U}=\sup_{\alpha>0.5}\sum_{j=1}^{n}I\left(Cr\{\tilde{X}_{ij}\geq\tilde{M}\}\right).$$

Now, to make a decision rule to in control or out of control regarding a statistical process, for imprecise observations, we apply a decision rule based on the concept of $\tilde{p}-value=[\tilde{p}^L-value,\tilde{p}^U-value]$ as follows [14:[

$$\tilde{p}^L-value=\min_{\omega\in\tilde{t}_i}2\min\left\{\sum_{j=0}^{\omega}\binom{j}{n}(0.5)^n,\sum_{j=\omega}^{n}\binom{j}{n}(0.5)^n\right\},$$

$$\tilde{p}^U-value=\max_{\omega\in\tilde{t}_i}2\min\left\{\sum_{j=0}^{\omega}\binom{j}{n}(0.5)^n,\sum_{j=\omega}^{n}\binom{j}{n}(0.5)^n\right\}.$$

Since the $\tilde{p}-value$ is an interval, at significance level $\delta=0.0027$, a degree of in-control process would be given as follows:

$$\varphi_\delta\left(\tilde{X}_{i1},\tilde{X}_{i2},\dots,\tilde{X}_{in}\right)=\begin{cases}1, & \tilde{p}^L>0.0027\\ \dfrac{\tilde{p}^U-0.0027}{\tilde{p}^U-\tilde{p}^L}, & \tilde{p}^L\le 0.0027\le\tilde{p}^U\\ 0, & \tilde{p}^U<0.0027\end{cases}$$

Clearly $\tilde{p}-value$ control chart has some properties that help the user to determine whether the point in a statistical process control is in control or not?

3.3.2 Fuzzy Wilcoxon signed-rank control chart based on credibility measure ranking method

Let $\tilde{X}_{i1},\tilde{X}_{i2},\dots,\tilde{X}_{in}$ be a random sample (sub-group) of the size $n>1$ observed from a continuous and a symmetric process with median $\tilde{M}$ (crisp or imprecise) at sampling instances $i=1,2,\dots,m.$ It is assumed that the IC process median is known to be equal to $\tilde{M}$. In each sub-group, the fuzzy Wilcoxon signed-rank (FSR) plotting statistics is computed as follows:

$$\tilde{W}_i=\{\tilde{W}_i{}^L,\tilde{W}_i{}^L+1,\dots,\tilde{W}_i{}^U\},\quad i=1,2,\dots,m,$$

in which,

$$\tilde{W}_i{}^L=\inf_{\alpha>0.5}\sum_{j=1}^{n}I\left(Cr\{\tilde{X}_{ij}\ge\tilde{M}\}\right)r\left(D_{2,\frac{1}{2}}(\tilde{X}_{ij},\tilde{M})\right),$$

$$\tilde{W}_i{}^U=\sup_{\alpha>0.5}\sum_{j=1}^{n}I\left(Cr\{\tilde{X}_{ij}\ge\tilde{M}\}\right)r\left(D_{2,\frac{1}{2}}(\tilde{X}_{ij},\tilde{M})\right).$$

In making decision to in control or out of control regarding a statistical process, we apply the $\tilde{p}-value$ concept as follows:

$$\tilde{p}^L-value=\min_{[\omega]\in\tilde{W}_i}2\min\left\{P\left(\tilde{W}\ge\omega\right),P\left(\tilde{W}\le\omega\right)\right\},$$

$$\tilde{p}^U-value=\max_{[\omega]\in\tilde{W}}2\min\left\{P\left(\tilde{W}\ge\omega\right),P\left(\tilde{W}\le\omega\right)\right\}.$$

For the similar fuzzy SN chart, at significance level $\delta=0.0027$, IC process degree is as follows:

$$\varphi_\delta\left(\tilde{X}_{i1},\tilde{X}_{i2},\dots,\tilde{X}_{in}\right)=\begin{cases}1, & \tilde{p}^L>0.0027\\ \dfrac{\tilde{p}^U-0.0027}{\tilde{p}^U-\tilde{p}^L}, & \tilde{p}^L\le 0.0027\le\tilde{p}^U\\ 0, & \tilde{p}^U<0.0027\end{cases}$$

Example 3 We illustrate the proposed approach by using a fuzzy data set related to inside piston diameter measurements in Example 2 The $\tilde{t}_i$ and $\tilde{W}_i$ statistics and $\tilde{p}-$values corresponding to their for each sub-group are calculated. The obtained results for 25 samples are shown in Table 3. This table indicates that the process is in control.

Remark 2 Kaya and Kahraman [11] considered the problem of monitoring location and scale parameters of a statistical process when sample observations are reported as imprecise. To accomplish this, they calculate fuzzy control limits based on three-sigma rule with sprite of Montgomery [17]. Then, the authors utilized defuzzification methods for imprecise control limits and thus change imprecise mean and range values in every sub-group of sample as crisp values. Finally, we applied classical methods for decision making. Note that, three-sigma rules are employed when the underlying distribution is normal, that unfortunately, this assumption is completely ignored in their approach. Hence, our proposed methods can be suitable to apply when there is no any knowledge regarding the underlying process distribution.

Remark 3 Senturk and Erginel [22] studied fuzzy $\bar{X}-R$ and $\bar{X}-S$ control charts with $\alpha-$cuts by using $\alpha-$ level fuzzy midrange transformation techniques. In their approach, $\alpha-$cuts of control limits were calculated based on three-sigma rule. Note that, three-sigma rules are used when process is normal, unfortunately, this assumption is ignored in their method.

Using the fuzzy data set of example 2, we compare the proposed control charts with respect to above mentioned control charts. The results comparing is listed Table 4. It is clear that all approaches produce similar results. The main advantages of our approach are simplicity in performance and computation, non usage defuzzification operators and flexibility in designing charts.

Table 3 The result of FSN and FSR control charts based on Cr measure ranking method

Sample number	$Cr\{\tilde{X}_{ij} \geq \tilde{M}\}\left(r\left(D_{2,\frac{1}{2}}(\tilde{X}_{ij},\tilde{M})\right)\right)$					$\tilde{t}_i$	$\tilde{W}_i$	$\tilde{P}$ For SN chart	$\tilde{P}$ For SR chart	Result
	1	2	3	4	5					
1	1(5)	0.75(1)	1(4)	0(2)	1(3)	{3,4}	{12,12.5,13}	1	[0.094,0.156]	In conrtol
2	0(2)	0(4)	0.625(1)	1(5)	1(3)	{1,2}	{3,3.5,4}	[0.375,1]	[0.156,0.438]	In conrtol
3	0(3)	1(5)	1(4)	1(2)	0.75(1)	{3,4}	{12,12.5,13}	1	[0.094,0.156]	In conrtol
4	0.75(2)	0(1)	0(3)	1(5)	1(4)	{2,3}	{9,9.5,…,11}	1	[0.219,0.406]	In conrtol
5	0(1)	1(2)	1(5)	0(3)	1(4)	{3}	{11}	1	0.219	In conrtol
6	1(4)	0(3)	0.125(1)	0(5)	0(2)	{1}	{4}	0.375	0.438	In conrtol
7	0(2)	1(5)	0(3)	0.33(1)	1(4)	{2}	{9}	1	0.406	In conrtol
8	0(4)	0.8751)	0(2)	1(5)	0(3)	{1,2}	{5,5.5,6}	[0.375,1]	[0.312,0.406]	In conrtol
9	1(4)	0(1)	1(5)	1(3)	1(2)	{4}	{14}	1	0.062	In conrtol
10	0.25(1)	0.5(2)	0(4)	1(5)	0(3)	{1}	{5}	0.375	0.312	In conrtol
11	0(4.5)	0.25(1)	0(4.5)	0(3)	0.75(2)	{1}	{2}	0.375	0.094	In conrtol
12	1(4)	0.5(2)	1(5)	0.33(1)	0(3)	{2}	{9}	1	0.406	In conrtol
13	0(5)	0.75(3)	0.25(1)	0(2)	1(4)	{1,2}	{3,3.5,…,7}	[0.375,1]	[0.156,0.5]	In conrtol
14	1(3)	0(5)	0(2)	0.33(1)	0(4)	{1}	{3}	0.375	0.156	In conrtol
15	1(4)	1(5)	0.25(1)	0.17(2)	1(3)	{3}	{12}	1	0.156	In conrtol
16	0.5(2)	0(5)	1(4)	0(1)	0(3)	{1}	{4}	0.375	0.219	In conrtol

17	0(1)	1(5)	0(4)	1(2)	1(3)	{3}	{10}	1	0.312	In conrtol
18	1(3)	1(4)	1(5)	0.83(2)	0.5(1)	{3,4}	{12,12.5,…,14}	1	[0.062,0.156]	In conrtol
19	0(5)	0.75(2)	0.875(3)	1(4)	0.125(1)	{1,2,3}	{4,4.5,…,9}	[0.375,1]	[0.219,0.406]	In conrtol
20	0.5(1)	1(3)	1(4)	1(5)	0.875(2)	{3,4}	{12,12.5,…,14}	1	[0.062,0.156]	In conrtol
21	0(4)	0.625(2)	1(5)	1(3)	0(1)	{3,4}	{8,8.5,…,10}	1	[0.312,0.5]	In conrtol
22	1(2)	0.375(1)	0(4)	1(3)	1(5)	{3}	{10}	1	0.312	In conrtol
23	1(4)	0(2)	0(1)	1(3)	1(5)	{3}	{12}	1	[0.624,1]	In conrtol
24	1(5)	1(3)	0(2)	0.33(1)	1(4)	{3}	{12}	1	0.156	In conrtol
25	0(4)	0(2)	0(1)	1(5)	1(3)	{2}	{8}	1	0.5	In conrtol

4 Sign chart to monitor variability process in fuzzy environment

In this section, we extend the sign chart for variability process to the case when the observation are imprecise rather than crisp.

Suppose that a sample of size $m\,(m \geq 20)$, $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_m$ is available from an IC process. Then at each sub-group i, a sample of size n is obtained from the process, and the pooled sample of size $m.n$ is obtained. In this situation, we have to combine imprecise observations as $\tilde{X}_{11}, \tilde{X}_{12}, ..., \tilde{X}_{1n}, \tilde{X}_{21}, \tilde{X}_{22}, ..., \tilde{X}_{mn}$. Then, observations in the pooled sample are ranked from smallest to the largest, and the ranks of observations is calculated. Hence, we apply distance between two fuzzy numbers to rank imprecise observations.

In order to obtain the fuzzy sign statistic at sample i, we need to be within third quartile $(\tilde{Q}_3)$ and the first quartile $(\tilde{Q}_1)$ estimated from process data when the process is IC. The steps of fuzzy sign statistic computation for control chart are as follows:

1. Set the arbitrary origin $\tilde{B}$,
2. Determine $D_{2,\frac{1}{2}}(\tilde{Q}_1, \tilde{B})$ and $D_{2,\frac{1}{2}}(\tilde{Q}_3, \tilde{B})$,
3. Determine $D_{2,\frac{1}{2}}(\tilde{X}_{ij}, \tilde{B})$ for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.
4. The fuzzy sign statistics are obtained by the following formula:

$$U_i = \sum_{i=1}^{m} \left[I\left(D_{2,\frac{1}{2}}(\tilde{Q}_1, \tilde{B}) > D_{2,\frac{1}{2}}(\tilde{X}_{ij}, \tilde{B}) \right) + I\left(D_{2,\frac{1}{2}}(\tilde{X}_{ij}, \tilde{B}) > D_{2,\frac{1}{2}}(\tilde{Q}_3, \tilde{B}) \right) \right], \tag{4.1}$$

for $i = 1, 2, ..., m$, where I is the indicator function.

The two sided chart signals OoC status when $U_i \geq d$ or $U_i \leq -d$.

Table 4 Comparison between various control charts for median in fuzzy environment

Sample number	SN control chart based on $D_{p,q}$ -distance method	SR control chart based on $D_{p,q}$ -distance method	Fuzzy control chart based on Kaya and Khahraman method	Fuzzy control chart based on Senturk and Erginel method
1	In conrtol	In conrtol	In conrtol	In conrtol
2	In conrtol	In conrtol	In conrtol	In conrtol
3	In conrtol	In conrtol	In conrtol	In conrtol
4	In conrtol	In conrtol	In conrtol	In conrtol
5	In conrtol	In conrtol	In conrtol	In conrtol
6	In conrtol	In conrtol	In conrtol	In conrtol
7	In conrtol	In conrtol	In conrtol	In conrtol
8	In conrtol	In conrtol	In conrtol	In conrtol
9	In conrtol	In conrtol	In conrtol	In conrtol
10	In conrtol	In conrtol	In conrtol	In conrtol
11	In conrtol	In conrtol	In conrtol	In conrtol
12	In conrtol	In conrtol	In conrtol	In conrtol
13	In conrtol	In conrtol	In conrtol	In conrtol
14	In conrtol	In conrtol	In conrtol	In conrtol
15	In conrtol	In conrtol	In conrtol	In conrtol
16	In conrtol	In conrtol	In conrtol	In conrtol
17	In conrtol	In conrtol	In conrtol	In conrtol
18	In conrtol	In conrtol	In conrtol	In conrtol
19	In conrtol	In conrtol	In conrtol	In conrtol
20	In conrtol	In conrtol	In conrtol	In conrtol
21	In conrtol	In conrtol	In conrtol	In conrtol
22	In conrtol	In conrtol	In conrtol	In conrtol
23	In conrtol	In conrtol	In conrtol	In conrtol
24	In conrtol	In conrtol	In conrtol	In conrtol
25	In conrtol	In conrtol	In conrtol	In conrtol

Example 4 We illustrate the operations of the synthetic control chart using fuzzy data set related to Table 1. The data set includes 25 samples each of five observations. Based on example 2, we conclude that this process is in control. Hence, we estimated $\tilde{Q}_1 = (73.991, 73.995, 73.996)_T$ and $\tilde{Q}_1 = (74.004, 74.008, 74.009)_T$ based on $mn = 125$ sample observations. To have an in control ARL equal to 32, the parameters of the synthetic control chart are $UCL = d = 4$ and $L = 1$ [12].

Table 5 illustrates the values of the fuzzy sign statistic U_i for 25 samples. We have constructed fuzzy sign chart and the CRL chart in Figure 3. At the beginning, we set CRL to be zero and increased its value by one for every conforming sample. Here, the sample is conforming, if $U_i < 4$. Otherwise, the sample is nonconforming. From Figure 3, it is clear that the first nonconforming sample occurs at time epoch 5 and the CRL at that time point is 5, which is greater than $L = 1$. Therefore, the process is assumed to be in-control and we reset CRL to be zero. The next nonconforming samples occurs at time epoch's 8, 12, 23 and 25 with CRL's equal to 3, 4, 11 and 2 respectively, again which is above L . Hence, we conclude that the process run is in-control.

It should be noted that Kaya and Kahraman approach [11] for these data has led to the same conclusion with respect to the variability process, see Table 6. Of course, the main advantages of the proposed approach in comparison with Kaya and Kahraman method are Simplicity and also there is no need to assume a particular distribution for the underlying process.

Table 5 Sign chart statistics to monitor variation

Sample number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
U_i	3	2	3	3	4	3	1	4	1	1	2	0	1	3	3	1	3	2	1	3	2	3	5	3	4

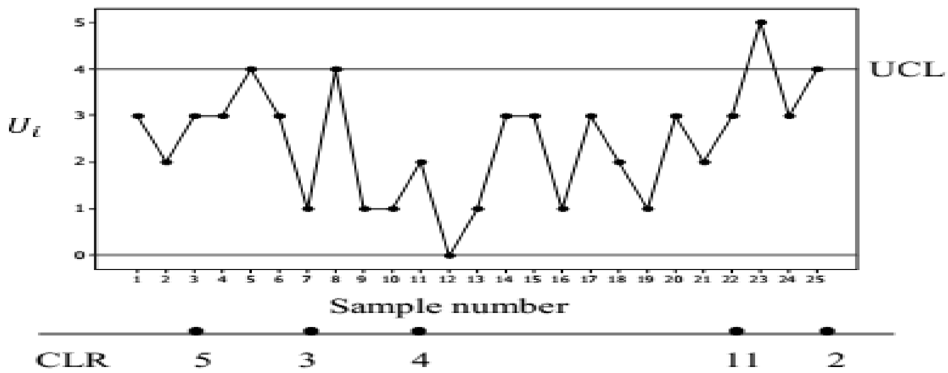

Fig. 3 Signed chart and fuzzy synthetic chart

Table 6 Comparison between control charts to monitor process variation in fuzzy environment

Sample number	SN control chart based on $D_{p,q}$ -distance method	Fuzzy control chart based on Kaya and Khahraman method	Sample numbe r	SN control chart based on $D_{p,q}$ -distance method	Fuzzy control chart based on Kaya and Khahraman method
1	In conrtol	In conrtol	14	In conrtol	In conrtol
2	In conrtol	In conrtol	15	In conrtol	In conrtol
3	In conrtol	In conrtol	16	In conrtol	In conrtol
4	In conrtol	In conrtol	17	In conrtol	In conrtol
5	In conrtol	In conrtol	18	In conrtol	In conrtol
6	In conrtol	In conrtol	19	In conrtol	In conrtol
7	In conrtol	In conrtol	20	In conrtol	In conrtol
8	In conrtol	In conrtol	21	In conrtol	In conrtol
9	In conrtol	In conrtol	22	In conrtol	In conrtol
10	In conrtol	In conrtol	23	In conrtol	In conrtol
11	In conrtol	In conrtol	24	In conrtol	In conrtol
12	In conrtol	In conrtol	25	In conrtol	In conrtol
13	In conrtol	In conrtol			

5 Conclusion

In this paper, we proposed a new approach to monitor median and the variability process, when there is no crisp data and suitable parametric assumptions. The advantages of our approach are as follows: simplicity in performance and computation, non usage defuzzification methods and flexibility in designing charts. We analyzed and evaluated the proposed control charts with numerical examples. We concluded that our approach in comparison with the same other methods can be efficient as compared with its fuzzy parametric control charts to identify the change location of the parameter and variability.

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