International Journal of Applied Operational Research
Vol. 3, No. 1, pp. 19-26, Winter 2013

Journal homepage: www.ijorlu.ir

A note on symmetric duality vector optimization problems

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Received: 2 May 2012 ; **Accepted:** 29 August 2012

Abstract In this paper, we establish weak and strong duality theorems for a pair of multi-objective symmetric dual problems. This removes several omissions in the paper "Symmetric and self duality in vector optimization problem, Applied Mathematics and Computation 183 (2006) 1121-1126".

Keywords Nonlinear Programming, Multiobjective Programming, Symmetric Duality, Efficient Solutions.

1 Introduction

Optimality conditions and duality have played an important role in the development of mathematical programming in single-objective as well as multi-objective problems. Several papers have appeared in the recent past on multi-objective problems. Arana et al. [1] discussed Kuhn-Tucker and Fitz-John type necessary and sufficient conditions and duality for a non-differentiable multiobjective problem.

Mangasarian [2] introduced the concept of second and higher-order duality for nonlinear programs. This motivated several authors in this field. Kim et al. [3] formulated a pair of second-order multiobjective symmetric dual programs and proved duality theorems. Yang and Hou [4] established duality results for second-order symmetric dual programs under invexity assumptions. In [5] and [6], Ahmad and Husain discussed Wolfe and Mond-Weir type multi-objective second-order symmetric dual programs with cone constraints under invexity and generalized invexity assumptions, respectively. Recently, Kailey and Gupta [7] studied non-differentiable second-order mixed symmetric duality with cone constraints under (F,ρ) convexity/psuedo-convexity assumptions. Agrawal et al. [8] established a strong duality theorem for Mond-Weir type multiobjective higher-order non-differentiable symmetric dual programs.

The symmetric dual problems in the above papers involve a single kernel function $f(x,y)$. Kassem [9] formulated the following pair of symmetric dual problems involving two kernel functions f and g:

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$$\begin{aligned} Min \quad & f(x,y)-(y^T\nabla_y(\lambda^T f)(x,y))e-(y^T\nabla_y(w^T g)(x,y)p)e \\ s.\ t. \quad & \nabla_y(\lambda^T f)(x,y)+p^T\nabla_y(w^T g)(x,y)\leqq 0, \qquad x,p\geqq 0, \\ & \lambda>0, \quad w>0, \quad \lambda^T e=1, \quad w^T e=1, \end{aligned}$$

and

$$\begin{aligned} Max \quad & f(u,v)-(u^T\nabla_x(\lambda^T f)(u,v))e-(u^T\nabla_x(w^T g)(u,v)p)e \\ s.\ t. \quad & \nabla_x(\lambda^T f)(u,v)+p^T\nabla_x(w^T g)(u,v)\geqq 0, \qquad v,p\geqq 0, \\ & \lambda>0, \quad w>0, \quad \lambda^T e=1, \quad w^T e=1, \end{aligned}$$

where $f: R^n \times R^n \to R^m$, $g: R^n \times R^n \to R^r$, $p \in R^n$, $\lambda \in R^m$, $w \in R^r$ and $e=(1,1,...,1)^T$ are of appropriate dimension.

It has been observed in [10] that involving two kernel functions as above has altered the dimensions of the terms in the objective function and the constraints. This has led to several omissions in the models as well as in the proofs of the duality theorems. For example, the first two terms in each objective function are vectors, while the third term is not a vector. Also, the **vector** $\nabla_y(w^T g)(x,y)$ has been assumed to be positivly definite [9].

In this note, we present symmetric dual multi-objective problems involving two kernel functions and establish weak, strong and converse duality theorems. It also serves to remove the omissions in [9].

2 Prerequisites

Let $K: R^n \to R^p$ be a twice differentiable function, $\nabla_x K(\nabla_y K)$ denote the $n\times p\,(m\times p)$ matrix of first order partial derivatives and $\nabla_{xy}K_i$ denote the $n\times m$ matrix of second order partial derivatives. All vectors shall be considered as column vectors. For two vectors a and b in R^n,

$(i).\ a \geqq b \Leftrightarrow a_i \geqq b_i\ (i=1,2,...,n);$

$(ii).\ a \geq b \Leftrightarrow a \geqq b,\ and\ a \neq b;$

$(iii).\ a > b \Leftrightarrow a_i > b_i\ (i=1,2,...,n).$

Consider the multi-objective optimization problem :

(P)

$$\begin{aligned} Min \quad & K(x)=\{K_1(x),K_2(x),...,K_p(x)\} \\ s.\ t. \quad & x \in X^\circ=\{x \in R^n : G(x)\leqq 0\}, \end{aligned}$$

where $G: R^n \to R^m$.

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Definition 1. [11] A point $\bar{x} \in X^{\circ}$ is said to be an efficient solution of (P) if there exists no $x \in X^{\circ}$ such that $K(x) \leq K(\bar{x})$.

3 Wolfe type Symmetric Duality

We now consider the following pair of Wolfe type symmetric multiobjective programming problems:

(WP)

$$Min \quad \{f(x,y) - (y^T \nabla_y (\lambda^T f(x,y)))e - (y^T (\nabla_{yy} (w^T g(x,y))p))e\}$$

$$s.t. \quad \nabla_y (\lambda^T f(x,y)) + \nabla_{yy} (w^T g(x,y))p \leqq 0, \tag{1}$$

$$\lambda^T e = 1, \lambda > 0, \tag{2}$$

$$x \geqq 0. \tag{3}$$

(WD)

$$Max \quad \{f(u,v) - (u^T \nabla_x (\lambda^T f(u,v)))e - (u^T (\nabla_{xx} (w^T g(u,v))q))e\}$$

$$s.t. \quad \nabla_x (\lambda^T f(u,v)) + \nabla_{xx} (w^T g(u,v))q \geqq 0, \tag{4}$$

$$\lambda^T e = 1, \lambda > 0, \tag{5}$$

$$v \geqq 0, \tag{6}$$

where

(i) $f : R^n \times R^m \to R^k$ is a twice differentiable function of x and y,

(ii) $g : R^n \times R^m \to R^r$ is a thrice differentiable function of x and y,

(iii) $\lambda \in R^k$, $w \in R^r$, $p \in R^m$, $q \in R^n$, $e = (1,1,...,1)^T \in R^k$,

If λ is fixed to be $\bar{\lambda}$ in problem (WD), then it will be denoted (WD)$_{\bar{\lambda}}$.

The duality results

Theorem 1. (Weak duality). Let (x, y, λ, w, p) be feasible for (WP) and (u, v, λ, w, q) be feasible for (WD). Let

(i) $f(.,v)$ be convex at u for fixed v,

(ii) $-f(x,.)$ be convex at y for fixed x,

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(iii) $(x-u)^T \nabla_{xx}(w^T g)(u,v)q \leqq 0$ *and* $(v-y)^T \nabla_{yy}(w^T g)(x,y)p \geqq 0.$

Then

$$f(u,v)-(u^T\nabla_x(\lambda^T f(u,v)))e-(u^T\nabla_{xx}(w^T g(u,v))q)e \ngeq$$
$$f(x,y)-(y^T\nabla_y(\lambda^T f(x,y)))e-(y^T\nabla_{yy}(w^T g(x,y))p)e).$$

Proof. Suppose to the contrary that

$$f(u,v)-(u^T\nabla_x(\lambda^T f(u,v)))e-(u^T\nabla_{xx}(w^T g(u,v))q)e \geq$$

$$f(x,y)-(y^T\nabla_y(\lambda^T f(x,y)))e-(y^T\nabla_{yy}(w^T g(x,y)p)e).$$

Since $\lambda > 0$ and $\lambda^T e = 1$, the above vector inequality implies

$$\lambda^T[f(u,v)-(u^T\nabla_x(\lambda^T f(u,v)))e-(u^T\nabla_{xx}(w^T g(u,v))q)e] >$$

$$\lambda^T[f(x,y)-(y^T\nabla_y(\lambda^T f(x,y)))e-(y^T\nabla_{yy}(w^T g(x,y))p)e]$$

or

$$\lambda^T f(u,v)-u^T\nabla_x(\lambda^T f(u,v))-u^T\nabla_{xx}(w^T g(u,v))q > \lambda^T f(x,y)-y^T\nabla_y(\lambda^T f(x,y)) \\ -y^T\nabla_{yy}(w^T g(x,y))p. \quad (7)$$

Now by convexity of $f(.,v)$, we have

$$f(x,v)-f(u,v) \geqq (x-u)^T \nabla_x f(u,v).$$

As $\lambda > 0$,

$$\lambda^T(f(x,v)-f(u,v)) \geqq (x-u)^T \nabla_x(\lambda^T f(u,v)). \quad (8)$$

Constraints (3) and (4) imply

$$x^T(\nabla_x(\lambda^T f(u,v))+\nabla_{xx}(w^T g(u,v))q) \geqq 0, \text{ or}$$

$$x^T\nabla_x(\lambda^T f(u,v)) \geqq -x^T\nabla_{xx}(w^T g(u,v))q$$

$$\geqq -u^T\nabla_{xx}(w^T g(u,v))q \quad \text{(by hypothesis (iii))}.$$

Therefore, inequality (8) yields

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$$\lambda^T f(x,v) - \lambda^T f(u,v) \geqq -u^T(\nabla_x(\lambda^T f(u,v)) + \nabla_{xx}(w^T g(u,v))q). \quad (9)$$

Similarly by hypothesis (ii), (iii) and constraints (1) and (6), we get

$$\lambda^T f(x,y) - \lambda^T f(x,v) \geqq y^T(\nabla_y(\lambda^T f(x,y)) + \nabla_{yy}(w^T g(x,y))p). \quad (10)$$

Adding (9) and (10), we obtain

$$\lambda^T f(x,y) - y^T\nabla_y(\lambda^T f(x,y)) - y^T\nabla_{yy}(w^T g(x,y))p \geqq \lambda^T f(u,v) - u^T\nabla_x(\lambda^T f(u,v)) - u^T\nabla_{xx}(w^T g(u,v))q,$$

which contradicts (7). This completes the proof.

Theorem 2. (Strong duality). Let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ be an efficient solution for (WP). Assume that

(i) $\nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y}))$ is nonsingular,
(ii) the set $\{\nabla_y f_i(\bar{x},\bar{y}), i = 1,2,\ldots,k\}$ is linearly independent, and
(iii) $\nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y}))\bar{p} \notin \text{span}\{\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})\} \setminus \{0\}$.

Then $(\bar{x},\bar{y},\bar{w},\bar{q}=0)$ is feasible for $(\text{WD})_{\bar{\lambda}}$, and the objective function values of $(\text{WP})_{\bar{\lambda}}$ and $(\text{WD})_{\bar{\lambda}}$ are equal. Also, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of $(\text{WP})_{\bar{\lambda}}$ and $(\text{WD})_{\bar{\lambda}}$, then $(\bar{x},\bar{y},\bar{w},\bar{q}=0)$ is an efficient solution for $(\text{WD})_{\bar{\lambda}}$.

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ is an efficient solution for (WP), by the Fritz John necessary optimality conditions [12], there exist $\bar{\alpha} \in R^k$, $\bar{\beta} \in R^m$, $\bar{\gamma} \in R^k, \bar{\eta} \in R$ *and* $\bar{\mu} \in R^n$, such that the following conditions are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$:

$$\nabla_x(\bar{\alpha}^T f(\bar{x},\bar{y})) + (\nabla_{yx}(\bar{\lambda}^T f(\bar{x},\bar{y})) + \nabla_x(\nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y}))\bar{p}))^T(\bar{\beta} - (\bar{\alpha}^T e)\bar{y}) - \bar{\mu} = 0, \quad (11)$$

$$\nabla_y f(\bar{x},\bar{y})(\bar{\alpha} - (\bar{\alpha}^T e)\bar{\lambda}) + (\nabla_{yy}(\bar{\lambda}^T f(\bar{x},\bar{y})) + \nabla_y(\nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y}))\bar{p}))(\bar{\beta} - (\bar{\alpha}^T e)\bar{y}) - (\bar{\alpha}^T e)\nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y}))\bar{p} = 0, \quad (12)$$

$$(\bar{\beta} - (\bar{\alpha}^T e)\bar{y})^T\nabla_y f(\bar{x},\bar{y}) + \bar{\eta}e^T - \bar{\gamma} = 0, \quad (13)$$

$$(\bar{\beta} - (\bar{\alpha}^T e)\bar{y})^T\nabla_w(\nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y}))\bar{p}) = 0, \quad (14)$$

$$(\bar{\beta} - (\bar{\alpha}^T e)\bar{y})^T\nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y})) = 0, \quad (15)$$

$$\bar{\beta}^T(\nabla_y(\bar{\lambda}^T f(\bar{x},\bar{y})) + \nabla_{yy}(\bar{w}^T g(\bar{x},\bar{y}))\bar{p}) = 0, \quad (16)$$

$$\bar{\lambda}^T\bar{\gamma} = 0, \quad (17)$$

$$\bar{x}^T\bar{\mu} = 0, \quad (18)$$

$$(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\mu}) \geq 0, \quad (19)$$

$$(\bar{\alpha},\bar{\beta},\bar{\eta},\bar{\gamma},\bar{\mu}) \neq 0. \quad (20)$$

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As $\bar{\lambda} > 0$, from (17) we conclude $\bar{\gamma} = 0$. By hypothesis (i), equation (15) implies

$$\bar{\beta} = (\bar{\alpha}^T e)\bar{y}. \tag{21}$$

Thus (13) yield $\bar{\eta} = 0$. Now suppose, $\bar{\alpha} = 0$. Then by relation (21), $\bar{\beta} = 0$ and so $\bar{\mu} = 0$ by (11), which contradicts $(\bar{\alpha}, \bar{\beta}, \bar{\eta}, \bar{\mu}) \neq 0$. Thus, $\bar{\alpha} \neq 0$, i.e., $\bar{\alpha} \geq 0$ or

$$\bar{\alpha}^T e > 0. \tag{22}$$

From (21) and (22),

$$\bar{y} = \frac{\bar{\beta}}{\bar{\alpha}^T e} \geqq 0.$$

Now, from (12) and (21), we have

$$\nabla_y f(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e)\bar{\lambda}) = (\bar{\alpha}^T e)\nabla_{yy}(\bar{w}^T g(\bar{x}, \bar{y}))\bar{p}. \tag{23}$$

Using hypothesis (iii), the above relation implies $(\bar{\alpha}^T e)\nabla_{yy}(\bar{w}^T g(\bar{x}, \bar{y}))\bar{p} = 0,$ which by (22) and hypothesis (i) imply

$$\bar{p} = 0. \tag{24}$$

Thus, (23) gives $\nabla_y f(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e)\bar{\lambda}) = 0.$
Since the set $\{\nabla_y f_i, i = 1, \ldots, k\}$ is linearly independent,

$$\bar{\alpha} = (\bar{\alpha}^T e)\bar{\lambda}. \tag{25}$$

Using (21), (22) and (25) in (11), we get

$$\nabla_x(\bar{\lambda}^T f(\bar{x}, \bar{y})) = \bar{\mu} \geqq 0.$$

Thus, $(\bar{x}, \bar{y}, \bar{w}, \bar{q} = 0)$ is a feasible solution for the dual problem $(\mathrm{WD})_{\bar{\lambda}}$. Further, from (16), (21), (22) and (24), we obtain $\bar{y}^T \nabla_y(\bar{\lambda}^T f(\bar{x}, \bar{y})) = 0$. Hence, $(\mathrm{WP})_{\bar{\lambda}}$ and $(\mathrm{WD})_{\bar{\lambda}}$ have equal objective function values. Now, suppose $(\bar{x}, \bar{y}, \bar{w}, \bar{q} = 0)$ is not an efficient solution for $(\mathrm{WD})_{\bar{\lambda}}$, then there exist (u, v, w, q) feasible for $(\mathrm{WD})_{\bar{\lambda}}$, such that

$$(f(u,v) - u^T \nabla_x(\lambda^T f(u,v))e - (u^T(\nabla_{xx}(w^T g(u,v))q))e) -$$
$$(f(\bar{x},\bar{y}) - \bar{x}^T \nabla_x(\bar{\lambda}^T f(\bar{x},\bar{y}))e - (\bar{x}^T(\nabla_{xx}(\bar{w}^T g(\bar{x},\bar{y}))\bar{q}))e) \geq 0.$$

Since

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$$\overline{p} = 0, \overline{q} = 0, \overline{x}^T \nabla_x (\overline{\lambda}^T f(\overline{x}, \overline{y})) = 0 \text{ and } \overline{y}^T \nabla_y (\overline{\lambda}^T f(\overline{x}, \overline{y})) = 0,$$
$$(f(u,v) - u^T \nabla_x (\lambda^T f(u,v))e - (u^T (\nabla_{xx} (w^T g(u,v))q))e) -$$
$$(f(\overline{x}, \overline{y}) - \overline{y}^T \nabla_y (\overline{\lambda}^T f(\overline{x}, \overline{y}))e - (\overline{y}^T (\nabla_{yy} (\overline{w}^T g(\overline{x}, \overline{y}))\overline{p}))e) \geq 0,$$

which contradicts the weak duality theorem. Hence $(\overline{x}, \overline{y}, \overline{w}, \overline{q} = 0)$ is an efficient solution for $(\text{WD})_{\overline{\lambda}}$.

Theorem 3. (Converse duality). Let $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{q})$ be an efficient solution for (WD). Assume that

(i) $\nabla_{xx} (\overline{w}^T g)(\overline{u}, \overline{v})$ is nonsingular,
(ii) the set $\{\nabla_x f_i (\overline{u}, \overline{v}), i = 1, \ldots, k\}$ is linearly independent, and
(iii) $\nabla_{xx} (\overline{w}^T g)(\overline{u}, \overline{v})\overline{q} \notin \text{span}\{\nabla_x f_1 (\overline{u}, \overline{v}), \ldots, \nabla_x f_k (\overline{u}, \overline{v})\} \setminus \{0\}$.

Then, $(\overline{u}, \overline{v}, \overline{w}, \overline{p} = 0)$ is feasible for $(\text{WP})_{\overline{\lambda}}$, and the objective function values of $(\text{WP})_{\overline{\lambda}}$ and (WD) are equal. Also, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of $(\text{WP})_{\overline{\lambda}}$ and $(\text{WD})_{\overline{\lambda}}$, then $(\overline{u}, \overline{v}, \overline{w}, \overline{p} = 0)$ is an efficient solution for $(\text{WP})_{\overline{\lambda}}$.

Proof. Follows on the lines of Theorem 2.

Acknowledgements

The second author is thankful to the MHRD, Government of India for providing financial support.

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