International Journal of Applied Operational Research
Vol. 3, No. 2, pp. 1-13, Spring 2013

Journal homepage: www.ijorlu.ir

A Study on Exponential Fuzzy Numbers Using α-Cuts

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Received: 2 October 2012 ; **Accepted:** 3 February 2013

Abstract In this study a new approach to rank exponential fuzzy numbers using α-cuts is established. The metric distance of the interval numbers is extended to exponential fuzzy numbers. By using the ranking of exponential fuzzy numbers and using α-cuts the critical path of a project network is solved and illustrated by numerical examples.
Key words Exponential Fuzzy Numbers, α-cuts, Metric Distance, Ranking, Critical Path.

1 Introduction

Various ranking procedures have been developed since 1976 when the theory of fuzzy sets were first introduced by Zadeh [1]. Ranking fuzzy numbers were first proposed by Jain [2] for decision making in fuzzy situations by representing the ill-defined quantity as a fuzzy set. Yager [3] proposed four indices which may be employed for the purpose of ordering fuzzy quantities in [0,1]. In Kaufmann and Gupta [4], an approach is presented for the ranking of fuzzy numbers. Campos and Gonzalez [5] proposed a subjective approach for ranking fuzzy numbers. Liou and Wang [6] developed a ranking method based on integral value index. Cheng [7] presented a method for ranking fuzzy numbers by using the distance method. Dubois and Prade [8] presented the mean value of a fuzzy number. Lee and Li [9] presented a comparison of fuzzy numbers based on the probability measure of fuzzy events. Delgado et al. [10] presented a procedure for ranking fuzzy numbers. Campos and Mu˜noz [11] presented a subjective approach for ranking fuzzy numbers. Kim and Park [12] presented a method of ranking fuzzy numbers with index of optimism. Yuan [13] presented a criterion for evaluating fuzzy ranking methods.

Heilpern [14] presented the expected value of a fuzzy number. Chen and Chen [15] presented a method for ranking generalized trapezoidal fuzzy numbers. Abbasbandy and Hajjari [16] introduced a new approach for ranking of trapezoidal fuzzy numbers based on the left and right spreads at some α-levels of trapezoidal fuzzy numbers.

Since late 1950s, Critical Path Method (CPM) techniques have become widely recognized as valuable tools for planning and scheduling complex projects. When the activity times in the project are deterministic and known, CPM has been demonstrated to be a useful method in managing projects in an efficient manner to meet the challenge [17]. There are many cases

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where the activity times may not be presented in a precise manner. To deal quantitatively with imprecise data, the program evaluation and review technique (PERT) [17,18] and Monte Carlo simulation [19] based on the probability theory can be employed. An alternative way to deal with imprecise data is to employ the concept of fuzziness [20], whereby the vague activity times can be represented by fuzzy numbers. Fuzzy numbers are used to describe uncertain activity durations, reflecting vagueness, imprecision and subjectivity in the estimation of them. There have been several attempts in the literature to apply fuzzy numbers to the critical path method since the late 1970s, and it has led to the development of fuzzy CPM [21-25]. In particular, problems of determining possible values of latest starting times and floats in networks with imprecise activity durations which are represented by fuzzy or interval numbers have attracted many researchers [21,22,23,26].

In [27], Dubois et al. extended the fuzzy arithmetic operations model to compute the latest starting time of each activity in a project network. In [28], Hapke et al. used fuzzy arithmetic operations to compute the earliest starting time for each activity in a project network. In [29], Yao et al. used signed distance ranking of fuzzy numbers to find critical path in a fuzzy network.

In [30] Chen et al. used defuzzification method to find possible critical paths in a fuzzy project network. Chanas and Zielinski [25] assume that the operation time of each activity can be represented as a crisp value, interval or a fuzzy number. Dubois et al. [31] assigns a different level of importance to each activity on a critical path for randomly chosen set of activities.

C.T. Chen et al. [32] proposed a method to deal with completion time management and the critical degrees of all activities for a project network.

Ranking of fuzzy numbers is an important component in the decision process. In this paper we present a new approach to rank exponential fuzzy numbers using α-cuts. First, we propose the metric distance of the interval numbers, and it is extended to exponential fuzzy numbers. Finally, the ranking of exponential fuzzy numbers using α-cuts is discussed and also finding critical path using the ranking is illustrated by a numerical example.

2 Definitions

Definition 2.1

A fuzzy set $\widetilde{A}$=(a,b,c,d) is an exponential fuzzy number if its membership function satisfies the following

$$\mu_{\widetilde{A}}(x) = \begin{cases} \exp\left[\dfrac{-(b-x)}{b-a}\right], & a \le x \le b \\ 1, & b \le x \le c \\ \exp\left[\dfrac{-(x-c)}{d-c}\right], & c \le x \le d \end{cases}$$

Here L(x)= $\exp\left[\frac{-(b-x)}{b-a}\right]$ and R(x) = $\exp\left[\frac{-(x-c)}{d-c}\right]$ are the left and right reference functions of exponential fuzzy number $\widetilde{A}$. Therefore, left inverse function of L(x) is given by $L^{-1}(\alpha)=b+(b-a)\log\alpha$. Similarly, the right inverse function of R(x) is given by $R^{-1}(\alpha)=c-(d-c)\log\alpha$. Hence, an exponential fuzzy number can be represented as $[\widetilde{A}]^{\alpha}=[b+(b-a)\log\alpha, c-(d-c)\log\alpha]$.

Definition 2.2
If $\widetilde{A}$ and $\widetilde{B}$ are two exponential fuzzy numbers then, $\widetilde{A}+\widetilde{B}=\widetilde{C}$ if and only if $[A]^{\alpha}+[B]^{\alpha}=[C]^{\alpha}$ and $\widetilde{A}.\widetilde{B}=\widetilde{C}$ if and only if $[A]^{\alpha}.[B]^{\alpha}=[C]^{\alpha}$
where [a,b]+[c,d] =[a+c, b+d]
[a,b]-[c,d] =[a-d, b-c]
and [a,b].[c.d]=[min{ac,ad,bc,bd},max{ac,ad,bc,bd}]

Definition 2.3
Let $\widetilde{A}=(a_1,b_1,c_1,d_1)$ and $\widetilde{B}=(a_2,b_2,c_2,d_2)$ be two exponential fuzzy numbers. Then
$[\widetilde{A}]^{\alpha}=[(b_1-a_1)\log\alpha+b_1, c_1-(d_1-c_1)\log\alpha]$ and
$[\widetilde{B}]^{\alpha}=[(b_2-a_2)\log\alpha+b_2, c_2-(d_2-c_2)\log\alpha]$. Then the distance between $[A]^{\alpha}\ and\ [B]^{\alpha}$ is denoted by $d_I^{(p)}[[A]^{\alpha},[B]^{\alpha}]$ such that $d_I^{(p)}[[A]^{\alpha},[B]^{\alpha}]=(D_I^{(p)}[[A]^{\alpha},[B]^{\alpha}])^{\frac{1}{p}}$, (1)
and $(D_I^{(p)}[[A]^{\alpha},[B]^{\alpha}])=$

$$\Big\|\Big(\big(c_1-(d_1-c_1)\log\alpha-(b_1-a_1)\log\alpha-b_1\big)x+(b_1-a_1)\log\alpha+b_1\Big)$$
$$-\Big(\big(c_2-(d_2-c_2)\log\alpha-(b_2-a_2)\log\alpha-b_2\big)x+(b_2-a_2)\log\alpha+b_2\Big)\Big\|_{L_p}^{p} \quad (2)$$

where $\|.\|$ is the usual norm in the L_p space on [0,1] (p>1).
Now let us show that our proposed distance is a metric distance on exponential fuzzy numbers.

3.1 Metric properties

The proposed distance satisfies the following metric properties:

1. $d_I^{(p)}[[A]^{\alpha},[B]^{\alpha}]\geq 0$
2. $[A]^{\alpha}=[B]^{\alpha}$ if and only if $d_I^{(p)}[[A]^{\alpha},[B]^{\alpha}]=0$
3. $d_I^{(p)}[[A]^{\alpha},[B]^{\alpha}]=d_I^{(p)}[[B]^{\alpha},[A]^{\alpha}]$
4. $d_I^{(p)}[[A]^{\alpha},[B]^{\alpha}]+d_I^{(p)}[[B]^{\alpha},[C]^{\alpha}]\geq d_I^{(p)}[[A]^{\alpha},[C]^{\alpha}]$

The proofs immediately follow from the properties of the L_p norm.

3.2 Other properties of $d_I^{(p)}$

The distance defined satisfies the following properties:

Proposition 1
If $\lambda \geq 0$, then

$$d_I^{(p)}([\lambda\{(b_1 - a_1)\log\alpha + b_1\}, \lambda\{c_1 - (d_1 - c_1)\log\alpha\}], [\lambda\{(b_2 - a_2)\log\alpha + b_2\},$$
$$\lambda\{c_2 - (d_2 - c_2)\log\alpha\}])$$
$$= |\lambda| d_I^{(p)}[[A]^\alpha, [B]^\alpha]$$

Proposition 2

$$d_I^{(p)}([\{(b_1 - a_1)\log\alpha + b_1\} + \lambda, \{c_1 - (d_1 - c_1)\log\alpha\} + \lambda], [\{(b_2 - a_2)\log\alpha + b_2\} + \lambda,$$
$$\{c_2 - (d_2 - c_2)\log\alpha\} + \lambda])$$
$$= d_I^{(p)}[[A]^\alpha, [B]^\alpha]$$

Proposition 3

$$d_I^{(p)}([\{(b_1 - a_1)\log\alpha + b_1\}, \{c_1 - (d_1 - c_1)\log\alpha\}] = \left|\{(b_1 - a_1)\log\alpha + b_1\} - \{c_1 - (d_1 - c_1)\log\alpha\}\right|$$

Proposition 4
If p=2, then

$$d_I^{(p)}[[A]^\alpha, [B]^\alpha] = \sqrt{\frac{1}{3}}\Big\{\big([(b_1 - a_1)\log\alpha + b_1] - [(b_2 - a_2)\log\alpha + b_2]\big)^2 +$$
$$\big(\{c_1 - (d_1 - c_1)\log\alpha\} - \{c_2 - (d_2 - c_2)\log\alpha\}\big)^2 +$$
$$\big([(b_1 - a_1)\log\alpha + b_1] - [(b_2 - a_2)\log\alpha + b_2]\big).\big(\{c_1 - (d_1 - c_1)\log\alpha\} - \{c_2 - (d_2 - c_2)\log\alpha\}\big)\Big\}$$

Proposition 5
If p=2, then

$$d_I^{(p)}\big([(b_1 - a_1)\log\alpha + b_1, c_1 - (d_1 - c_1)\log\alpha], (b_2 - a_2)\log\alpha + b_2\big) =$$
$$\sqrt{\frac{1}{3}}\Big\{\big([(b_1 - a_1)\log\alpha + b_1] - [(b_2 - a_2)\log\alpha + b_2]\big)^2 + \big(\{c_1 - (d_1 - c_1)\log\alpha\} - [(b_2 - a_2)\log\alpha + b_2]\big)^2 +$$
$$\big([(b_1 - a_1)\log\alpha + b_1] - [(b_2 - a_2)\log\alpha + b_2]\big).\big(\{c_1 - (d_1 - c_1)\log\alpha\} - [(b_2 - a_2)\log\alpha + b_2]\big)\Big\}$$

Proposition 6
$\dfrac{(b_1 - a_1)\log\alpha + b_1 + c_1 - (d_1 - c_1)\log\alpha}{2}$ is the nearest number to
$[(b_1 - a_1)\log\alpha + b_1, c_1 - (d_1 - c_1)\log\alpha]$ by using this metric.

Proposition 7

If p=2, then $d_I^{(p)}\left([(b_1-a_1)\log\alpha+b_1, c_1-(d_1-c_1)\log\alpha],0\right)=$

$$\sqrt{\frac{1}{3}}\left\{\left((b_1-a_1)\log\alpha+b_1\right)^2+\left(c_1-(d_1-c_1)\log\alpha\right)^2+\left([(b_1-a_1)\log\alpha+b_1]\right).\left(\{c_1-(d_1-c_1)\log\alpha\}\right)\right\}$$

The proof of propositions 1,2,3,4,5,6 and 7 are obvious .

Proposition 8

$$\begin{aligned}&d_I^{(p)}([\{(b_1-a_1)\log\alpha+b_1\},\{c_1-(d_1-c_1)\log\alpha\}]+[\{(b_2-a_2)\log\alpha+b_2\},\{c_2-(d_2-c_2)\log\alpha\}],\\&[\{(b_1-a_1)\log\alpha+b_1\},\{c_1-(d_1-c_1)\log\alpha\}]+[\{(b_3-a_3)\log\alpha+b_3\},\{c_3-(d_3-c_3)\log\alpha\}])\\&=d_I^{(p)}([\{(b_2-a_2)\log\alpha+b_2\},\{c_2-(d_2-c_2)\log\alpha\}],[\{(b_3-a_3)\log\alpha+b_3\},\{c_3-(d_3-c_3)\log\alpha\}])\end{aligned}$$

Proof.

$$\begin{aligned}&d_I^{(p)}([\{(b_1-a_1)\log\alpha+b_1\},\{c_1-(d_1-c_1)\log\alpha\}]+[\{(b_2-a_2)\log\alpha+b_2\},\{c_2-(d_2-c_2)\log\alpha\}],\\&\quad[\{(b_1-a_1)\log\alpha+b_1\},\{c_1-(d_1-c_1)\log\alpha\}]+[\{(b_3-a_3)\log\alpha+b_3\},\{c_3-(d_3-c_3)\log\alpha\}])\end{aligned}$$

$$\begin{aligned}=\ &d_I^{(p)}\Big(\big[(b_1-a_1)\log\alpha+b_1+(b_2-a_2)\log\alpha+b_2, c_1-(d_1-c_1)\log\alpha+c_2-(d_2-c_2)\log\alpha\big],\\&\big[(b_1-a_1)\log\alpha+b_1+(b_3-a_3)\log\alpha+b_3, c_1-(d_1-c_1)\log\alpha+c_3-(d_3-c_3)\log\alpha\big]\Big)\end{aligned}$$

$$\begin{aligned}=\ &\Bigg(\int_0^1\Big[\big(c_1-(d_1-c_1)\log\alpha+c_2-(d_2-c_2)\log\alpha-(b_1-a_1)\log\alpha-b_1-(b_2-a_2)\log\alpha-b_2\big)x\\&+(b_1-a_1)\log\alpha+b_1+(b_2-a_2)\log\alpha+b_2\Big]-\Big[\big(c_1-(d_1-c_1)\log\alpha+c_3-(d_3-c_3)\log\alpha-\\&(b_1-a_1)\log\alpha-b_1-(b_3-a_3)\log\alpha-b_3\big)x\quad+(b_1-a_1)\log\alpha+b_1+(b_3-a_3)\log\alpha+b_3\Big]^p\,dx\Bigg)^{\frac{1}{p}}\end{aligned}$$

$$\begin{aligned}=\ &\Bigg\{\int_0^1\Big\{\Big[\big(c_2-(d_2-c_2)\log\alpha-(b_2-a_2)\log\alpha-b_2\big)x+(b_2-a_2)\log\alpha-b_2\Big]-\\&-\Big[\big(c_3-(d_3-c_3)\log\alpha-(b_3-a_3)\log\alpha-b_3\big)x+(b_3-a_3)\log\alpha-b_3\Big]\Big\}^p\,dx\Bigg\}^{\frac{1}{p}}\end{aligned}$$

$$=$$

$$d_I^{(p)}([\{(b_2-a_2)\log\alpha+b_2\},\{c_2-(d_2-c_2)\log\alpha\}],[\{(b_3-a_3)\log\alpha+b_3\},\{c_3-(d_3-c_3)\log\alpha\}])$$

Definition 3.3

If $\left[(b_1-a_1)\log\alpha+b_1, c_1-(d_1-c_1)\log\alpha\right]$ & $\left[(b_2-a_2)\log\alpha+b_2, c_2-(d_2-c_2)\log\alpha\right]$ are two intervals and M is a crisp number then by (1) & (2), we have

$$d_I^{(p)}\left(\left[(b_1-a_1)\log\alpha+b_1, c_1-(d_1-c_1)\log\alpha\right].\left[(b_2-a_2)\log\alpha+b_2, c_2-(d_2-c_2)\log\alpha\right], M\right)$$

$$=\left(D_I^{(p)}\left([(b_1-a_1)\log\alpha+b_1, c_1-(d_1-c_1)\log\alpha].[(b_2-a_2)\log\alpha+b_2, c_2-(d_2-c_2)\log\alpha], M\right)\right)^{\frac{1}{p}}$$

where

$$D_I^{(p)}\left([(b_1-a_1)\log\alpha+b_1, c_1-(d_1-c_1)\log\alpha].[(b_2-a_2)\log\alpha+b_2, c_2-(d_2-c_2)\log\alpha], M\right)$$

$$=\int_0^1 \begin{array}{l} D_I^{(p)}[(b_1-a_1)\log\alpha+b_1, c_1-(d_1-c_1)\log\alpha]. \\ \left((c_2-(d_2-c_2)\log\alpha-(b_2-a_2)\log\alpha-b_2\right)x+(b_2-a_2)\log\alpha+b_2, M\right)dx \end{array}$$

4 Metric for exponential fuzzy number

α - distance between 2 exponential fuzzy numbers $\widetilde{A}\,\&\,\widetilde{B}$ can be defined as

$d_F^{(p)}(\widetilde{A},\widetilde{B})=\left(D_F^{(p)}(\widetilde{A},\widetilde{B})\right)^{\frac{1}{p}}$ such that $D_F^{(p)}(\widetilde{A},\widetilde{B})=\sup_{\alpha\in[0,1]} D_I^{(p)}\left([A]^\alpha,[B]^\alpha\right)$

4.1 Properties of α - distance

a. $d_F^{(p)}(\widetilde{A},\widetilde{B})\geq 0$

b. $\widetilde{A}=\widetilde{B}$ if and only if $d_F^{(p)}(\widetilde{A},\widetilde{B})=0$

c. $d_F^{(p)}(\widetilde{A},\widetilde{B})=d_F^{(p)}(\widetilde{B},\widetilde{A})$

d. $d_F^{(p)}(\widetilde{A},\widetilde{B})+d_F^{(p)}(\widetilde{B},\widetilde{C})\geq d_F^{(p)}(\widetilde{A},\widetilde{C})$

Proof

a. We know that $D_I^{(p)}\left([A]^\alpha,[B]^\alpha\right)\geq 0$

$$\Rightarrow D_F^{(p)}(\widetilde{A},\widetilde{B})\geq 0 \Rightarrow d_F^{(p)}(\widetilde{A},\widetilde{B})\geq 0$$

b. Clearly $\widetilde{A}=\widetilde{B}$ if and only if $[A]^\alpha=[B]^\alpha\ \forall\alpha\in[0,1]$.

But $[A]^\alpha=[B]^\alpha$ if and only if $D_F^{(p)}(\widetilde{A},\widetilde{B})=0$ *i.e.*, $d_F^{(p)}(\widetilde{A},\widetilde{B})=0$

c. $d_F^{(p)}(\widetilde{A},\widetilde{B})=\left(D_F^{(p)}(\widetilde{A},\widetilde{B})\right)^{\frac{1}{p}}$

$$=\left(\sup_{\alpha\in[0,1]} D_I^{(p)}\left([A]^\alpha,[B]^\alpha\right)\right)^{\frac{1}{p}}$$

$$=\left(\sup_{\alpha\in[0,1]} D_I^{(p)}\left([B]^\alpha,[A]^\alpha\right)\right)^{\frac{1}{p}}$$

$$=\left(D_F^{(p)}(\widetilde{B},\widetilde{A})\right)^{\frac{1}{p}}$$

$$=d_F^{(p)}(\widetilde{B},\widetilde{A})$$

d. $d_F^{(p)}(\widetilde{A},\widetilde{C})=\left(D_F^{(p)}(\widetilde{A},\widetilde{C})\right)^{\frac{1}{p}}$

$$=\left(\sup_{\alpha\in[0,1]} D_I^{(p)}\left([A]^\alpha,[C]^\alpha\right)\right)^{\frac{1}{p}}$$
$$=\sup_{\alpha\in[0,1]}\left(D_I^{(p)}\left([A]^\alpha,[C]^\alpha\right)\right)^{\frac{1}{p}}$$
$$=\sup_{\alpha\in[0,1]}\left(d_I^{(p)}\left([A]^\alpha,[C]^\alpha\right)\right)$$
$$\leq\sup_{\alpha\in[0,1]}\left(d_I^{(p)}\left([A]^\alpha,[B]^\alpha\right)\right)+\left(d_I^{(p)}\left([B]^\alpha,[C]^\alpha\right)\right)$$
$$\leq\sup_{\alpha\in[0,1]}\left(d_I^{(p)}\left([A]^\alpha,[B]^\alpha\right)\right)+\sup_{\alpha\in[0,1]}\left(d_I^{(p)}\left([B]^\alpha,[C]^\alpha\right)\right)$$
$$=d_F^{(p)}(\widetilde{A},\widetilde{B})+d_F^{(p)}(\widetilde{B},\widetilde{C})$$

4.2 Other properties of α - distance

Proposition 9
If $\lambda\geq 0$, then $d_F^{(p)}\left(\lambda\widetilde{A},\lambda\widetilde{B}\right)=|\lambda|d_F^{(p)}(\widetilde{A},\widetilde{B})$

Proposition 10
$d_F^{(p)}\left(\widetilde{A}+\lambda,\widetilde{B}+\lambda\right)=d_F^{(p)}(\widetilde{A},\widetilde{B})$

Proposition 11
$d_F^{(p)}\left(\widetilde{A}+\widetilde{C},\widetilde{A}+\widetilde{B}\right)=d_F^{(p)}(\widetilde{C},\widetilde{B})$

Proof
Using Proposition 8,

$$d_F^{(p)}\left(\widetilde{A}+\widetilde{C},\widetilde{A}+\widetilde{B}\right)=\left(d_F^{(p)}\left(\widetilde{A}+\widetilde{C},\widetilde{A}+\widetilde{B}\right)\right)^{\frac{1}{p}}$$
$$=\left(\sup_{\alpha\in[0,1]} D_I^{(p)}\left([A]^\alpha+[C]^\alpha,[A]^\alpha+[B]^\alpha\right)\right)^{\frac{1}{p}}$$
$$=\left(\sup_{\alpha\in[0,1]} D_I^{(p)}\left([C]^\alpha,[B]^\alpha\right)\right)^{\frac{1}{p}}$$
$$=\left(D_F^{(p)}\left(\widetilde{C},\widetilde{B}\right)\right)^{\frac{1}{p}}$$
$$=\left(d_f^{(p)}\left(\widetilde{C},\widetilde{B}\right)\right)$$

Proposition 12
$d_F^{(p)}((b_1-a_1)\log\alpha+b_1,c_1-(d_1-c_1)\log\alpha)=\left|(b_1-a_1)\log\alpha+b_1-c_1+(d_1-c_1)\log\alpha\right|$

Proposition 13
$d_F^{(p)}([A]^\alpha,[B]^\alpha)=d_I^{(p)}([A]^\alpha,[B]^\alpha)$

Definition 4.3
If $\widetilde{A}$ & $\widetilde{B}$ are two exponential fuzzy numbers and M is a crisp number then

$$d_F^{(p)}(\widetilde{A}.\widetilde{B},M) = \left(D_F^{(p)}(\widetilde{A}.\widetilde{B},M)\right)^{\frac{1}{p}}$$

where

$$D_F^{(p)}(\widetilde{A}.\widetilde{B},M) = \sup_{\alpha\in[0,1]} D_I^{(p)}([A]^\alpha.[B]^\alpha,M)d\alpha$$

Then

$$D_F^{(p)}(\widetilde{A}.\widetilde{B},M) = \sup_{\alpha\in[0,1]} \int_0^1 D_I^{(p)}([A]^\alpha.((B_l(\alpha)-(B_r(\alpha))x+(B_r(\alpha)),M)dx$$

If $\widetilde{B}=1$ then $d_F^{(p)}(\widetilde{A}.\widetilde{B},M) = d_F^{(p)}(\widetilde{A},M)$

5 The proposed method for ranking exponential fuzzy numbers using α -distance

In this section, we propose a new approach for ranking exponential fuzzy numbers $\widetilde{A}$ & $\widetilde{B}$ based on the α-distance. To rank two fuzzy numbers $\widetilde{A}$ & $\widetilde{B}$, if sup(supp($\widetilde{A}$))< inf(supp($\widetilde{B}$)), then $\widetilde{A}<\widetilde{B}$ and the degree of $\widetilde{A}<\widetilde{B}$ is 1 and the degree of $\widetilde{A}>\widetilde{B}$ is 0. Hence, we consider two fuzzy numbers $\widetilde{A}$ & $\widetilde{B}$ such that supp($\widetilde{A}$)$\cap$supp($\widetilde{B}$)$\neq\phi$. In our method we compare the degree of distance between the exponential fuzzy numbers and max(M) and min(m), where

$$M \geq \max(\text{supp}\,\widetilde{A} \cup \text{supp}\,\widetilde{B})$$

and

$$m \leq \min(\text{supp}\,\widetilde{A} \cup \text{supp}\,\widetilde{B})$$

Let the degree of distance between the exponential fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ and max(M) and min(m) be denoted by $\zeta_d^{(p)}(\widetilde{A},M)$ and $\zeta_d^{(p)}(\widetilde{A},m)$, which are given by

$$\zeta_d^{(p)}(\widetilde{A},M) = \frac{d_F^{(p)}(\widetilde{A},M)}{d_F^{(p)}(\widetilde{A},M)+d_F^{(p)}(\widetilde{A},m)}$$

$$\zeta_d^{(p)}(\widetilde{A},m) = \frac{d_F^{(p)}(\widetilde{A},m)}{d_F^{(p)}(\widetilde{A},M)+d_F^{(p)}(\widetilde{A},m)}$$

Proposition 14

$$\zeta_d^{(p)}(\widetilde{A},M) + \zeta_d^{(p)}(\widetilde{A},m) = 1$$

Proposition 15

$\zeta_d^{(p)}(\widetilde{A},M) \geq \zeta_d^{(p)}(\widetilde{B},M)$ if and only if $\zeta_d^{(p)}(\widetilde{A},m) \geq \zeta_d^{(p)}(\widetilde{B},m)$

Proposition 16
If M'>M and $\zeta_d^{(p)}(\widetilde{A},M)\geq\zeta_d^{(p)}(\widetilde{B},M)$ then $\zeta_d^{(p)}(\widetilde{A},M')\geq\zeta_d^{(p)}(\widetilde{B},M)$

Proposition 17
If m'<m and $\zeta_d^{(p)}(\widetilde{A},m)\leq\zeta_d^{(p)}(\widetilde{B},m)$ then $\zeta_d^{(p)}(\widetilde{A},m')\leq\zeta_d^{(p)}(\widetilde{B},m)$

5.1 Ranking procedure

The ranking procedure is defined as follows:

$$\text{(i) } \widetilde{A}\prec\widetilde{B}\Leftrightarrow \begin{cases}\zeta_d^{(p)}(\widetilde{A},M)\geq\zeta_d^{(p)}(\widetilde{B},M)\\ \qquad or\\ \zeta_d^{(p)}(\widetilde{A},m)\leq\zeta_d^{(p)}(\widetilde{B},m)\end{cases}$$

such that the degree of ranking is defined as

$$\zeta_{(\lambda M+(1-\lambda)m)}^{(\widetilde{A}<\widetilde{B})}=\frac{\lambda..\zeta_d^{(p)}(\widetilde{A},M)}{\zeta_d^{(p)}(\widetilde{A},M)+\zeta_d^{(p)}(\widetilde{B},M)}+\frac{(1-\lambda)..\zeta_d^{(p)}(\widetilde{B},M)}{\zeta_d^{(p)}(\widetilde{A},m)+\zeta_d^{(p)}(\widetilde{B},m)}\qquad where\ \lambda\in[0,1]$$

$$\text{(ii) } \widetilde{A}=\widetilde{B}\Leftrightarrow d_F^{(p)}(\widetilde{A},\widetilde{B})=0$$

Then, the degree of ranking is defined as $\zeta_{(\lambda M+(1-\lambda)m)}^{(\widetilde{A}=\widetilde{B})}=1$ and $\zeta_{(\lambda M+(1-\lambda)m)}^{(\widetilde{A}>\widetilde{B})}=\zeta_{(\lambda M+(1-\lambda)m)}^{(\widetilde{A}<\widetilde{B})}=0$

$$\text{(iii) } \widetilde{A}\approx\widetilde{B}\Leftrightarrow \begin{cases}\zeta_d^{(p)}(\widetilde{A},M)=\zeta_d^{(p)}(\widetilde{B},M)=\frac{1}{2}\\ \qquad \&\\ \zeta_d^{(p)}(\widetilde{A},m)=\zeta_d^{(p)}(\widetilde{B},m)=\frac{1}{2}\end{cases}$$

In this case the degree of ranking is defined as $\zeta_{(\lambda M+(1-\lambda)m)}^{(\widetilde{A}=\widetilde{B})}=\zeta_{(\lambda M+(1-\lambda)m)}^{(\widetilde{A}>\widetilde{B})}=\zeta_{(\lambda M+(1-\lambda)m)}^{(\widetilde{A}<\widetilde{B})}=1/2$

Hence, in this method, for ranking n exponential fuzzy numbers $\widetilde{A}_1,....\widetilde{A}_n$, we compare the degree of distance between the exponential fuzzy numbers and max(M) and min(m).

If we add $\widetilde{C}$ to the set the value of M and m may change but the ranking will not change.

6 Calculating fuzzy time values and critical path in a fuzzy project network

A fuzzy project network is an acyclic digraph, where the vertices represent events, and the directed edges represent the activities, to be performed in a project. Formally, a fuzzy project network is represented by N=(V,A,T). Let V={$v_1,v_2,\ldots v_n$} be a set of vertices, where v_1 and v_n are the start and final events of the project and each v_i belongs to some path from v_1 to v_n. Let $A\subset VxV$ be a set of directed edge $a_{ij}=(v_i,v_j)$, that represents the activities to be performed in the project. Activity a_{ij} is then represented by one and only one arrow starting with an event

v_i and ending with event v_j. For each activity a_{ij}, a fuzzy number $\widetilde{t}_{ij} \in T$ is defined, where $\widetilde{t}_{ij}$ is the fuzzy time required for the completion of a_{ij}. A critical path is a longest path from v_1 to v_n and an activity a_{ij} on the critical path is called a critical activity. Let $\widetilde{E}_i$ & $\widetilde{L}_i$ be the earliest event time for event I, respectively. Let $\widetilde{E}_j$ & $\widetilde{L}j$ be the earliest event time, and the latest event time for event j, respectively. Let $D_j = \{i / i \in V \; and \; a_{ij} \in A\}$ be a set of events obtained from event $j \in V$ and i<j. We then obtain E_j using the following equations

$$\tilde{E}_j = \max[\tilde{E}_i \oplus \tilde{t}_{ij}], i \in D_j, where\, \tilde{E}_1 = 0$$

Similarly, let $H_i = \{j/j \in V$ and $a_{ij} \in A\}$ be a set of events obtained from event $i \in V$ and i<j. We then obtain $\widetilde{L}_i$ using the following equations.

$$\tilde{L}_i = \min[\tilde{L}_j \Theta \tilde{t}_{ij}], j \in H_i, where\, \tilde{L}_n = \tilde{E}_n$$

The interval $[\widetilde{E}_i, \widetilde{L}_j]$ is the time during which a_{ij} must be completed. When the earliest fuzzy event time and latest fuzzy event time have been obtained, we can calculate the total float of each activity. For activity i-j in a fuzzy network, the total float of the activity i-j can be computed as follows;

$$\widetilde{T}_{ij} = \widetilde{L}_j \Theta \widetilde{E}_i \Theta \widetilde{t}_{ij}$$

Hence, we can obtain the earliest fuzzy event time, latest event time and the total float of every activity by using the above equations.

We defuzzify the total float of each activity and find the critical path of the network using the above ranking procedure.

6.1 Numerical example

The following figure shows the network representation of a fuzzy project network, where all the time durations are exponential fuzzy numbers.

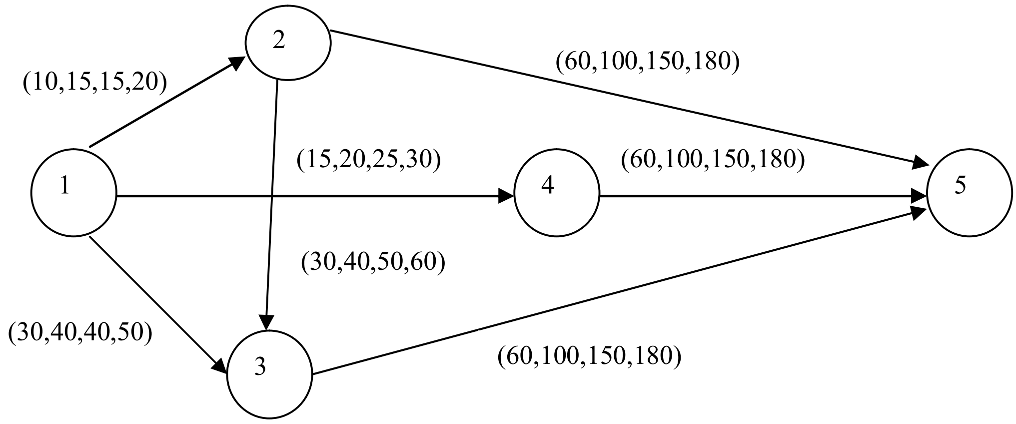

Fig. 1 (a,b,c,d) representation

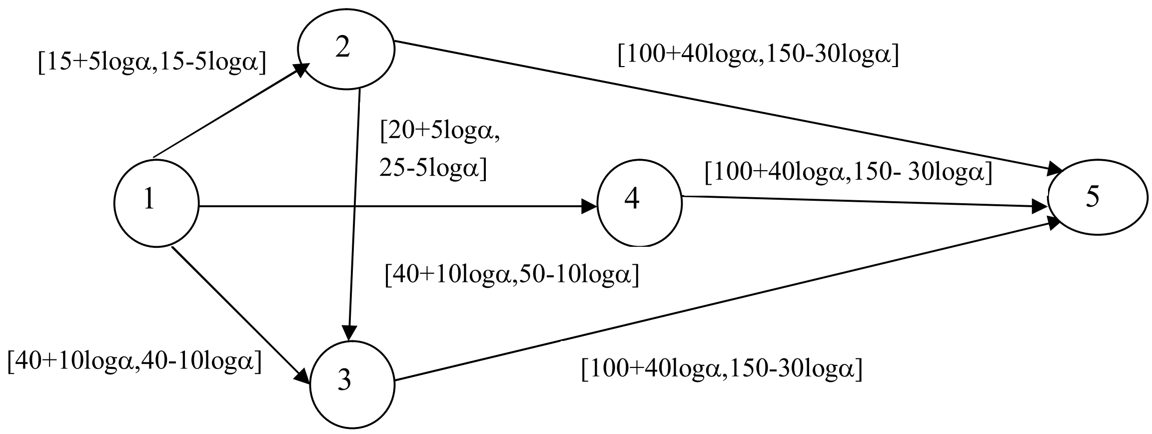

Fig. 2 α-cut representation

Table 1 Total float for each activity

Activity	Duration	Earliest Start	Latest Finish	Total float
1-2	[15+5logα,15-5logα]	[0,0]	[-45+90logα,75-75logα]	[-60+95logα,60-70logα]
1-3	[40+10logα,40-10logα]	[0,0]	[5+85logα,115-85logα]	[-35+95logα,75-75logα]
1-4	[20+5logα, 25-5logα]	[0,0]	[5+85logα,115-85logα]	[-20+80logα,95-90logα]
2-3	[40+10logα,50-10logα]	[15+5logα,15-5logα]	[5+85logα,115-85logα]	[-60+100logα,60-70logα]
2-5	[100+40logα,150-30logα]	[15+5logα,15-5logα]	[155+55logα,215-45logα]	[-10+90logα,100-90logα]
3-5	[100+40logα,150-30logα]	[55+15logα,65-15logα]	[155+55logα,215-45logα]	[-60+100logα,50-100logα]
4-5	[100+40logα,150-30logα]	[20+5logα,25-5logα]	[155+55logα,215-45logα]	[-20+90logα,95-90logα]

Table 2 Total float for the paths

Paths	Total Float
1-2-3-5	[-180+295logα,170-240logα]
1-2-5	[-70+185logα,160-160logα]
1-3-5	[-95+195logα,125-175logα]
1-4-5	[-40+170logα,190-180logα]

By using the above ranking procedure, it can be observed that the critical path is 1-2-3-5.

7 Conclusions

In this paper, a metric distance on exponential fuzzy numbers was proposed. Ranking procedure was defined based on the degree of distance between the exponential fuzzy numbers a max(M) and min(m). Even if another exponential fuzzy number is added the ranking of exponential fuzzy numbers will not change even if M and m may change. This is a very useful property in industrial problems. Also, we have illustrated by finding the critical path in a project network where all the numbers are considered as exponential fuzzy numbers.

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