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A Fuzzy Approach to Mean-CDaR Portfolio Optimization

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Abstract This paper develops a bi-objective portfolio selection problem that maximizes returns and minimizes a risk measure called conditional Drawdown (CDD). The drawdown measures include the maximal Drawdown and Average Drawdown as its limiting case. The CDD family of risk functional is similar to conditional value at Risk (CVaR). In this paper, the fuzzy method has been used to solve the bi-objectives model. The relevance of the proposed model is illustrated by a real life portfolio selection.

Keywords Portfolio Selection, Conditional Drawdown, Bi-Objectives Programming, Fuzzy Method.

1 Introduction

Portfolio optimization is the process of allocating budget between asset and managing the assets within it. The Modern portfolio theory has been proposed by Markowitz [1] that considers return and risk for portfolio selection problem. In the original Markowitz problem Variance is used as risk measure. The portfolio optimization model by Markowitz is a quadratic programming problem and has difficulties. Quadratic programming problems are more difficult to solve than linear problems and for real life portfolio selection problem, the size of covariance matrix for solving problem is very large and difficult to estimate. There are many attempts trying to reduce the difficulties of portfolio selection problem such as: Konno and Yamazaki [2], Michalowski and Ogryczak [3], Rockafellar and uryaser [4], Kellere et al [5], Mansini et al [6], Chiodi et al [7] and papahristodoulou and Dotzauer [8].

The CDD is related to value at Risk (VaR) and conditional value at Risk (CVaR).

Definition 1.1. (value at Risk (VaR)). Let k be a random variable and let F be its distribution function, that is $F(h) = P\{k \leq h\}$. Let $F(w) = \min\{h : F(h) \geq w\}$ be it's αVaR as the $\alpha - quantile$ of k.

$$\alpha - VaR(k) = F^{-1}(\alpha) \quad (1)$$

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Definition 1.2. (CVaR).Let $x \in X \subset R^N$ be a decision vector representing a portfolio, $y \in T \subset R^N$ be a vector representing the future value of a number of variables, and for each x denote by $\psi(x,0)$ the distribution function of the loss $Z = f(x,y)$ i.e.

$$\psi(x,\alpha) = P\{y \mid f(x,y) \le \alpha\} \tag{2}$$

Given $\alpha > 0$, the $\alpha - CVaR$ of loss associated with x is the mean of the $\alpha - tail$ distribution of the loss function, that is the mean of distribution function $\psi_\alpha(x,0)$ defined by

$$\psi_\alpha(x,a) = \begin{cases} 0 & if\ a < a_\alpha(x) \\ \dfrac{\psi(x,a)-\alpha}{1-\alpha} & if\ a \ge a_\alpha(x) \end{cases} \tag{3}$$

and $a_\alpha(x)$ is the $\alpha - VaR$ of the loss associated with X.

The CDD family of risk functional is similar to conditional value at Risk (CVaR).

There are many attempts trying to use multi-objectives for portfolio selection problems. We refer to Muhlemann et al [9], Levary and Avery [10], Tamiz et al [11], Deng et al [12], Li and Xu [13].

In the above models, crisp mathematics used for solving them. The concept of fuzzy was introduced by Zadeh in 1965. Bellman and Zadeh[14] developed fuzzy programming. Another step on this concept was taken by Li and Hwang [15] and zimmermann [16]. There are some multi-objective models that use Fuzzy mathematic to solve or to consider ambiguousness, such as watada [17], Inuihuchi and Ramik [18], Parra et al [19], Wang and Zhu [20], Ghazanfar ahari et al [21], Sadjadi et al [22].

This paper develops a portfolio selection problem that uses CDD measure as risk measure.. We contribute a bi-objectives portfolio selection problem and we use Fuzzy mathematics method to solve it. This is the first time in portfolio selection problem research that used CDD risk measure in multi-objective model and this is the first time that uses Zimmerman fuzzy method to solve multi-objective portfolio selection problem. The organization of this paper is as follows. Second section introduces CDD measure. Third section develops bi-objectives portfolio selection problem. In section four, the use of fuzzy mathematics is introduced for solving bi-objectives problems. In section five, empirical study in real life portfolio is performed and in final section conclusion expressed.

2 Drawdown measure

There are different measures and functions that are used in portfolio selection problem as risk measure. In this section we introduce Drawdown measure as a risk measure. This risk measure considers different sequences of portfolio losses.

Let portfolio be optimized within time interval [0,T], and let W(t) be portfolio value at time $t \in [0,T]$. The portfolio drawdown defined by $(\max_{T\in[0,T]} W(T) - W(t))/W(t)$.

We suppose that the time interval [0,T] is divided into N subintervals $[t_{l-1}, t_k]$, k=1,…,N. there are some points like $\{t_0 = 0, t_1, t, ..., t_N = T\}$. We suppose that the rates of returns of assets determined by random vector $r(t_k) = (r_1(t_k), r_2(t_k), ..., r_m(t_k))$ at time t_k for k=1,…,N. $P_i(t_k)$

is i^{th} asset's prices per share at time t_k, therefore the rate of return defined by $r_i(t_k) = \frac{P_i(t_k) - P_i(t_{k-1})}{P_i(t_{k-1})}$.

$x(t_k)$ is vector of weights that defined as follow. $x(t_k) = (x_0(t_k), x_1(t_k), x_2(t_k), ..., x_m(t_k))$. The constraint for budget defined as follow:

$$\sum_{i=0}^{m} x_i(t_k) = 1 \tag{4}$$

The portfolio rate of return at moment (t_k) defined as follow:

$$r_k x(x(t_k)) = r(t_k).x(t_k) = \sum_{i=0}^{m} r_i(t_k) x_i(t_k) \tag{5}$$

Sequence of assets rates of returns don’t account for VaR, CVaR and variance. Therefore they can't make sense in a dynamic case.

Portfolio optimization with drawdown constraints in continuous dynamics is considered by Cvitanic and Karatzas [23], Grossman and Zhou [24].

2.1 Absolute Drawdown

This section introduces Absolute Drawdown (AD) that is applied to a sample path of the uncompounded cumulative portfolio rate of return.

The uncompounded cumulative portfolio rate of return at time moment t_k is:

$$w_k(x(t_k)) = \begin{cases} 0 & k = 0 \\ \sum_{l=1}^{k} r_l^{(p)}(x(t_l)) & k = 1, ..., N \end{cases} \tag{6}$$

The AD is a vectorial-functional depending on the sample path based on Chekhlov et al. [25]:

$$AD(W) = \xi = (\xi_1, ..., \xi_N). \qquad \xi_k = \max_{0 \le j \le k} \{W_j\} - W_k \tag{7}$$

AD has some advantages instead of DD that discussed in Chekhlov et al. [25].

2.2 Maximum, Average and conditional Drawdown

This section presents the notion of maximum Drawdown (MaxDD), Average Drawdown (AvDD) and conditional Drawdown (CDD).

On the time interval [0,T].Divided into N subintervals $[t_{k-1}, t_k]$, k=1,…,N. with $t_0 = 0$ and $t_N = T$, maximum and average drawdown functional are defined as follow:

$$MaxDD(w) = \max_{1 \le k \le N} \{\xi_k\} \tag{8}$$

$$AvDD_{(w)} = \frac{1}{N}\sum_{k=1}^{N}\xi_k \tag{9}$$

We introduce function $\pi_{\xi(s)}$ as follow:

$$\pi_{\xi(s)} = \frac{1}{N}\sum_{k=1}^{N} I_{\{\xi_k \le s\}} \tag{10}$$

That:

$$I_{\{c \le s\}} = \begin{cases} 1 & c \le s \\ 0 & c \le s \end{cases} \quad c \in R \tag{11}$$

The inverse function to (10) is defined

$$\pi^{-1}_{\xi(\alpha)} = \begin{cases} \inf\left\{s \mid \pi_{\xi(s)} \ge \alpha\right\} & \alpha \in (0,1] \\ 0 & \alpha = 0 \end{cases} \tag{12}$$

$\pi^{-1}_{\xi(0)} = 0$ because all $\xi_k, k = 1,...,N$ are nonnegative.
Let $\xi(\alpha)$ be a threshold that (1-α) 100% of drawdown exceed this threshold. By definition:

$$\xi_{(\alpha)} = \pi^{-1}_{4(\alpha)} \tag{13}$$

By counting $(1-\alpha)*100\%$ of worst drawdown, then $\pi_\varepsilon(\xi(\alpha)) = \pi_\varepsilon(\pi^{-1}_\varepsilon(\alpha)) = \alpha$ and the CVaR of $\xi_k, k = 1,...,N$ is defined as the mean of the worst $(1-\alpha)*100\%$ drawdown [2]:

$$CVaR_{(\varepsilon)} = (\frac{\pi_\varepsilon(\xi(\alpha)) - \alpha}{1-\alpha})\xi_{(\alpha)} + \frac{1}{(1-\alpha)N}\sum \xi_k \tag{14}$$

2.3 Computation of the CDD

If we have drawdown of shares then computation ofthe $\alpha - CDD$ islike $\text{CVaR}_\alpha(\varepsilon)$:

$$CVaR_\alpha(\varepsilon) = \underset{y,z}{Min}\ y + \frac{1}{(1-\alpha)N}\sum_{k=1}^{N} z_k \tag{15}$$

s.t.

$$z_k \ge \xi_k - y \tag{16}$$

$$z_k \ge 0 \qquad k = 1,...,N \tag{17}$$

y equal to $\xi_{(\alpha)}$ if $\pi_\xi(\xi_{(\alpha)}) > \alpha$

If we have rate of return of shares ($r_1,...,r_N$) then the CDD functional, $\Delta_\alpha(w)$, is as follow:

$$\Delta_\alpha(w) = \underset{u,y,z}{Min}\; y + \frac{1}{(1-\alpha)N}\sum_{k=1}^{N} z_k \tag{18}$$

s.t.

$$z_k \geq u_k - y \tag{19}$$

$$u_k \geq u_{k-1} - r_k \qquad u_0 = 0 \tag{20}$$

$$z_k \geq 0, u_k \geq 0, \qquad k = 1,...,N \tag{21}$$

z_k and u_k are auxiliary variables and y equal to $\xi_{(\alpha)}$ if $\pi_\varepsilon(\xi_{(\alpha)}) > \alpha$

3 Fuzzy programming for Bi-objective portfolio selection problem

In this section, we propose a bi-objectives portfolio selection problem. We use the drawdown measure as a risk measure.
Zimmermann [16] developed Fuzzy modeling as follows:

$$Max \quad f(x) = Cx \tag{22}$$

s.t.

$$Ax \leq b \tag{23}$$

$$x \geq 0 \tag{24}$$

b_0 is a level that objective function of above formulation is greater than or equal to it:

$$Find \quad x \tag{25}$$

s.t.

$$Cx \geq b_0 \tag{26}$$

$$Ax \leq b \tag{27}$$

$$x \geq 0 \tag{28}$$

Zimmermann proposed membership function as follows:

$$\widetilde{A}^0(x) = \begin{cases} 1 & if \quad Cx \geq b_0 \\ 1 - \dfrac{b_0 - Cx}{P_0} & if \quad b_0 - P_0 \leq Cx \leq b_0 \\ 0 & if \quad Cx \leq b_0 - P_0 \end{cases}$$

$$\widetilde{A}^i(x) = \begin{cases} 1 & if \quad (Ax)_i \geq b_i \\ 1 - \dfrac{(Ax)_i - b_i}{P_0} & if \quad b_i \leq (Ax)_i \leq b_i + P_i \\ 0 & if \quad (Ax)_i \geq b_i + P_i \end{cases}$$

For $i = 1,...,m$ $(Ax)_i$ refer to the i[th] row of Ax with an auxility variable that is called λ the model formulated as follows:

$$Max \quad \lambda \tag{29}$$

s.t.

$$Cx \geq b_0 - (1-\lambda)P_0 \tag{30}$$

$$(Ax)_i \leq b_i + (1-\lambda)P_i \qquad i = 1,...,m \tag{31}$$

$$x \geq 0 \qquad \lambda \in [0,1] \tag{32}$$

We define Z^0 and Z^1 as follow:

$$b_0 - P_0 = Z^0 = \inf f = Min\ Cx \tag{33}$$

s.t.

$$(Ax)_i \leq b_i \tag{34}$$

$$x \geq 0 \tag{35}$$

and

$$b_0 = Z^1 = \sup f = \max Cx \tag{36}$$

s.t.

$$(Ax)_i \leq b_i + P_i \tag{37}$$

$$x \geq 0 \tag{38}$$

This model has two criteria such as portfolio rate of return that maximized and risk that minimized. Because the objective functions of multi-objective programming are more than one, it is difficult to reach a certain solution for each objective function. Therefore, it needs to make an accord plan that makes each function as optimal as possible. Fuzzy programming approach can turn the multi-target model to a single one.

The Mean-Drawdown model formulated as follow:

$$Max \quad E_w(w(T,w,x)) = \sum_{j=1}^{k} P_j w_j N(x) \tag{39}$$

$$Min \quad f(DD_x(w(x))) \tag{40}$$

s.t.

$$\sum_{i=0}^{m} x_i = 1 \tag{41}$$

$$x_i \geq 0 \qquad i = 0,...,m \qquad J = 1,...,k \tag{42}$$

i=0 refer to risk free asset like cash and P_j refer to the probability of event w_j. J=1,…,k refer to different scenario.

$$w_{jk}(x) = \sum_{l=1}^{k} r_{jl}^{P}(x) = \sum_{i=1}^{m} \sum_{l=1}^{k} r_{ij}(t_l) x_i \tag{43}$$

We can use different drawdown measure instead of (23) likes absolute drawdown, MaxDD, AvDD, and conditional drawdown.

To solve this model with fuzzy mathematic, we should solve the following sub-model. We use CDD in this model as risk measure.

$$R^{+} = \begin{cases} Max \quad E_w(w(T,w,x))\sum_{j=1}^{k} P_j W_j N(x) \\ s.t. \\ \quad \sum_{i=0}^{m} x_i = 1, \\ \quad x_i \geq 0. \end{cases} \tag{44}$$

$$R^{-} = \begin{cases} Min \quad E_w(w(T,w,x)) = \sum P_j W_j N(x) \\ s.t. \\ \quad \sum_{i=0}^{m} x_i = 1, \\ \quad x_i \geq 0. \end{cases} \tag{45}$$

The next step is to solve follow sub-models:

$$CDD^{+} = \begin{cases} Max \quad CVaR_{(\varepsilon)} \\ s.t. \\ \quad \sum_{i=0}^{m} x_i = 1, \\ \quad x_i \geq 0. \end{cases} \tag{46}$$

$$CDD^{-} = \begin{cases} Min \quad CVaR_{(\varepsilon)} \\ s.t. \\ \quad \sum_{i=0}^{m} x_i = 1, \\ \quad x_i \geq 0. \end{cases} \tag{47}$$

As mentioned before $CVaR_{\varepsilon}$ =CDD if we use absolute drawdown to measure CVaR. If we use rate of return to calculate CDD we should compute $\Delta_x(w_{(x)})$ as conditional drawdown.

In next step we use following single objective to solve above bi-objective portfolio selection problem.

$$Max \quad \lambda \tag{48}$$

$s.t.$

$$E_{(w)} - (R^{+} - R^{-})\lambda \geq R^{-} \tag{49}$$

$$CDD_{(\varepsilon)} - (CDD^{+} - CDD^{-})\lambda \leq CDD^{+} \tag{50}$$

$$\sum_{i=0}^{m} {}_{i} = 1 \tag{51}$$

$$\lambda \geq 0 \qquad x_i \geq 0 \tag{52}$$

In above formulation we can use different drawdown measure as risk measure.

4 Computational results

In this section, we present the results of an empirical study of Fuzzy multi-objectives model for portfolio selection problem. In this Numerical study, we use one year historical data for 10 financial assets. We divide one year to12 subinterval and we compute rate of returns monthly. In these Numerical results we compute CDD with an appropriate level ($\alpha = 0.8$). It means optimizing over the 20% of the worst drawdowns. In this section we use sterling ratio to compare results with together the sterling ratio calculate as follow:

$$SR = \frac{Annual\ Portfolio\ Return}{Average\ Largest\ Drawdown + 10\%}$$

We solve our model in case of one sample path at first we solveproblems (27) to (29) and (30) to (32). This problems give us R^{+} and R^{-} and we solve problems (33) to (35) and (36) to (38) this problems give us CDD^{+} and CDD^{-}. In CDD^{+} the result is unbounded solution to overcome this problem we add an extra constraint to bind the solution. The extra constraint is as follow: $CVaR_{(\varepsilon)}$<K. K is based on decision maker idea in this problem, we assume that k=0.8.

$R^{+}, R^{-}, CDD^{+} and\ CDD^{-}$ Are as follow:

$R^{+} = 0.2633350$
$R^{-} = -0.4274071$
$CDD^{+} = 0.8$
$CDD^{-} = 0.08142672$

We use additional constraint to better understand the results. We add $\lambda = 0$, $\lambda = 0.5$, $\lambda = 0.7$, $\lambda = 0.8$, $\lambda = 0.85$ and $\lambda = 1$ then we run model for each of them and we calculate SR for each of them and we run our model without additional constraint that show the optimal solution. We summarize the results in table 1.

Table 1. summary of solving the problem

λ	Rate of return (annual)	0.8-CDD	Sterling ratio
0	0.03354383	0.09980401	0.167883668
0.5	0.03353429	0.09980401	0.167835921
0.7	0.05611237	0.09766652	0.283873921
0.8	0.1251866	0.1397407	0.522175
0.85	0.1597237	0.1562646	0.623276488
0.9	0.1942608	0.1732526	0.710920225
1	0.263335	0.4044825	0.521990356

The optimal solution is in $\lambda = 0.9$ and as we show the optimal solution has the best sterling ratio. We show the efficient frontier in Figure1:

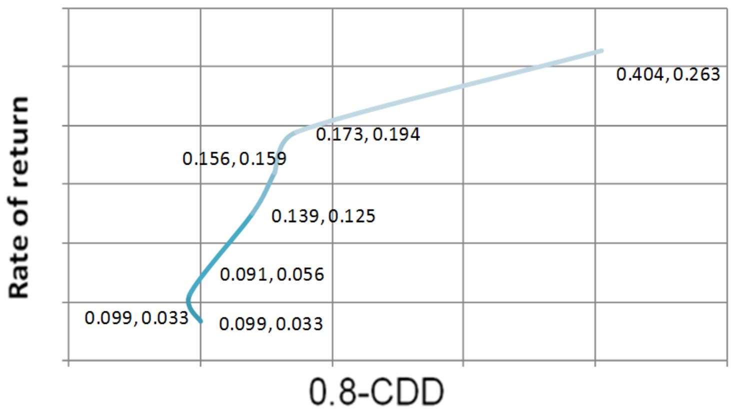

Fig. 1 Efficient Frontier

These results are based on one sample path and with 0.8-CDD with different λ.

5 Conclusions

In this paper, we introduced a bi-objective portfolio selection problem and we used the Fuzzy method to reformulate and solved that model. In single objective portfolio selection problem there is a constraint that limit return or risk but in this model there isn't that kind of constraint and in this model we can take different solution based on change of λ that show fuzziness of objectives. Our studies indicate that the optimal λ show better sterling ratio that is a scale that show how much the results are good. For future result we propose the use of robust optimization in fuzzy modeling because in fuzzy we consider ambiguity and in robust optimization we consider uncertainty with these two approach we can consider both ambiguity and uncertainty.

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