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Optimization of solution Kadomtsev-Petviashvili equation by using hompotopy methods

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Abstract In this paper, the Kadomtsev-Petviashvili equation is solved by using the Adomian's decomposition method , modified Adomian's decomposition method , variational iteration method , modified variational iteration method, homotopy perturbation method, modified homotopy perturbation method and homotopy analysis method. The existence and uniqueness of the solution and convergence of the proposed methods are proved in details. A numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords Kadomtsev-Petviashvili equation, Adomian decomposition method (ADM), Modified Adomian decomposition method (MADM), Variational iteration method (VIM), Modified variational iteration method (MVIM), Homotopy perturbation method (HPM), Modified homotopy perturbation method (MHPM), Homotopy analysis method (HAM).

1 Introduction

In 1970, Kadomtsev and Petviashvili [1] generalized the KDV equation to two space variables and formulated the well-known Kadmotsev-Petviashvili equation to provide an explanation of the general weakly dispersive waves [2-10]. In this work, we develope the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve this equation as follows:

$$u_t + \mu(x,t)u_x + \frac{1}{2}\sigma^2(x,t)u_{xx} - v(x,t)u + s(x,t) = 0. \tag{1}$$

With the initial condition:

$$u(x,0) = g(x). \tag{2}$$

Where $g(x)$, $\mu(x,t)$, $\sigma(x,t)$, $v(x,t)$ and $s(x,t)$ are known functions.

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1). In section 3 we prove the existence , uniqueness of the solution and convergence of the proposed methods. Finally, the numerical example is shown in section 4.

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In order to obtain an approximate solution of Eq.(1), let us integrate one time Eq.(1) with respect to t using the initial condition we obtain,

$$u(x,t) = G(x,t) - \int_0^t F_1(u(x,\tau)\, d\tau - \int_0^t F_2(u(x,\tau))\, d\tau + \int_0^t F_3(u(x,\tau))\, d\tau, \tag{3}$$

where,

$$G(x,t) = g(x) - \int_0^t s(x,\tau)\, d\tau,$$
$$F_1(u(x,t)) = \mu(x,t)u_x(x,t),$$
$$F_2(u(x,t)) = \frac{1}{2}\sigma^2(x,t)u_{xx}(x,t),$$
$$F_3(u(x,t)) = v(x,t)u(x,t).$$

In Eq.(3), we assume $G(x,t)$ is bounded for all x,t in $J = [0,T](T \in \mathrm{R})$.
The terms $F_1(u(x,t))$, $F_2(u(x,t))$ and $F_3(u(x,t))$ are Lipschitz continuous with $| F_1(u) - F_1(u^*) | \le L_1 | u - u^* |$, $| F_2(u) - F_2(u^*) | \le L_2 | u - u^* |$ and $| F_3(u) - F_3(u^*) | \le L_3 | u - u^* |$.

2 The iterative methods
2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = f, \tag{4}$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and f is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(4), and using the given conditions we obtain

$$u(x,t) = z(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{5}$$

where the function $z(x)$ represents the terms arising from integrating the source term f. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where $A_n, n \ge 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]]_{\lambda=0}. \tag{7}$$

The first Adomian polynomials (introduced in [11,12,13]) are:

$$
\begin{aligned}
A_0 &= G_1(u_0),\\
A_1 &= u_1 G_1'(u_0),\\
A_2 &= u_2 G_1'(u_0) + \frac{1}{2!}u_1^2 G_1''(u_0),\\
A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!}u_1^3 G_1'''(u_0),\ldots
\end{aligned} \tag{8}
$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of $u(x,t)$ in (4) as the following series,

$$
u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{9}
$$

where, the components $u_0(x,t), u_1(x,t),\ldots$ which can be determined recursively

$$
\begin{aligned}
&u_0(x,t) = G(x,t),\\
&u_1(x,t) = -\int_0^t A_0(x,t)\,dt - \int_0^t B_0(x,t)\,dt + \int_0^t L_0(x,t)\,dt,\\
&\vdots\\
&u_{n+1}(x,t) = -\int_0^t A_n(x,t)\,dt - \int_0^t B_n(x,t)\,dt + \int_0^t L_n(x,t)\,dt, \quad n \ge 0.
\end{aligned} \tag{10}
$$

Substituting (8) into (10) leads to the determination of the components of $u(x,t)$.

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [14]. The modified forms was established on the assumption that the function $G(x,t)$ can be divided into two parts, namely $G_1(x,t)$ and $G_2(x,t)$. Under this assumption we set

$$
G(x,t) = G_1(x,t) + G_2(x,t). \tag{11}
$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part G_1 be assigned to the zeroth component u_0, whereas the remaining part G_2 be combined with the other terms given in (11) to define u_1. Consequently, the modified recursive relation

$$
\begin{aligned}
&u_0(x,t) = G_1(x,t),\\
&u_1(x,t) = G_2(x,t) - L^{-1}(Ru_0) - L^{-1}(A_0),\\
&\vdots\\
&u_{n+1}(x,t) = -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \ge 1,
\end{aligned} \tag{12}
$$

was developed.

To obtain the approximation solution of Eq.(1), according to the MADM, we can write the iterative formula (12) as follows:

$$\begin{aligned}
&u_0(x,t)=G_1(x,t),\\
&u_1(x,t)=G_2(x,t)-\int_0^t A_0(x,t)\,dt-\int_0^t B_0(x,t)\,dt+\int_0^t L_0(x,t)\,dt,\\
&\vdots\\
&u_{n+1}(x,t)=-\int_0^t A_n(x,t)\,dt-\int_0^t B_n(x,t)dt+r\int_0^t L_n(x,t)\,dt,\quad n\geq 1.
\end{aligned}\tag{13}$$

The operators $F_i(u(x,t))$ $(i=1,2,3)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$F_1(u)=\sum_{i=0}^{\infty}A_i,$$
$$F_2(u)=\sum_{i=0}^{\infty}B_i,$$
$$F_3(u)=\sum_{i=0}^{\infty}L_i.$$

where A_i, B_i and L_i are the Adomian polynomials.
Also, we can use the following formula for the Adomian polynomials [15]:

$$\begin{aligned}
A_n&=F_1(s_n)-\sum_{i=0}^{n-1}A_i,\\
B_n&=F_2(s_n)-\sum_{i=0}^{n-1}B_i,\\
L_n&=F_3(s_n)-\sum_{i=0}^{n-1}L_i.
\end{aligned}\tag{14}$$

Where $s_n=\sum_{i=0}^{n}u_i(x,t)$ is the partial sum.

2.2 Description of the VIM and MVIM

In the VIM [16-23], it has been considered the following nonlinear differential equation:

$$Lu+Nu=G,\tag{15}$$

where L is a linear operator, N is a nonlinear operator and G is a known analytical function. In this case, the functions u_n may be determined recursively by

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda(x,\tau)\{L(u_n(x,\tau))+N(u_n(x,\tau))-G(x,\tau)\}d\tau,\quad n\geq 0,\tag{16}$$

where λ is a general Lagrange multiplier which can be computed using the variational theory. Here the function $u_n(x,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(x,t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{17}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq.(1), according to the VIM, we can write iteration formula (16) as follows:

$$\begin{aligned}&u_{n+1}(x,t) = u_n(x,t) + L_t^{-1}(\lambda[u_n(x,t) - G(x,t) + \int_0^t F_1(u_n(x,t))\, dt \\ &+ \int_0^t F_2(u_n(x,t))dt - \int_0^t F_2(u_n(x,t))\, dt]), n \geq 0.\end{aligned} \tag{18}$$

Where,

$$L_t^{-1}(.) = \int_0^t (.)\, d\tau.$$

To find the optimal λ, we proceed as

$$\begin{aligned}&\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta L_t^{-1}(\lambda[u_n(x,t) - G(x,t) + \int_0^t F_1(u_n(x,t))\, dt \\ &+ \int_0^t F_2(u_n(x,t))dt - \int_0^t F_3(u_n(x,t))\, dt]).\end{aligned} \tag{19}$$

From Eq.(19), the stationary conditions can be obtained as follows:

$$\lambda' = 0 \text{ and } 1 + \lambda = 0.$$

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in (18), the following iteration formula is obtained.

$$\begin{aligned}&u_0(x,t) = G(x,t), \\ &u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}(u_n(x,t) - G(x,t) + \int_0^t F_1(u_n(x,t))\, dt \\ &+ \int_0^t F_2(u_n(x,t))dt - \int_0^t F_3(u_n(x,t))\, dt), n \geq 0.\end{aligned} \tag{20}$$

To obtain the approximation solution of Eq.(1), based on the MVIM [24,25], we can write the following iteration formula:

$$u_0(x,t) = G(x,t),$$
$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}(\int_0^t F_1(u_n(x,t) - u_{n-1}(x,t))\, dt \tag{21}$$
$$+\int_0^t F_2(u_n(x,t) - u_{n-1}(x,t))dt - \int_0^t F_3(u_n(x,t) - u_{n-1}(x,t))\, dt), n \geq 0.$$

Relations (20) and (21) will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

2.3 Description of the HAM

Consider $N[u] = 0,$ where N is a nonlinear operator, $u(x,t)$ is an unknown function and x is an independent variable. let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x,t)] = 0$ when $r(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] = \hat{H}[\phi(x,t;q);u_0(x,t),H_1(x,t),h,q]. \tag{22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (22) to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q);u_0(x,t),H_1(x,t),h,q] = 0, \tag{23}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \tag{24}$$

When $q = 0$, the zero-order deformation Eq.(24) becomes

$$\phi(x;0) = u_0(x,t), \tag{25}$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq.(24) is equivalent to

$$\phi(x,t;1) = u(x,t). \tag{26}$$

Thus, according to (26) and (26), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [26-29].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{27}$$

where,

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \Big|_{q=0} .$$

Let the initial guess $u_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series (27) of $\phi(x,t;q)$ converges at $q=1$, then, we have under these assumptions the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{28}$$

From Eq.(27), we can write Eq.(24) as follows

$$\begin{gathered}(1-q)L[\phi(x,t,q) - u_0(x,t)] = (1-q)L[\sum_{m=1}^{\infty} u_m(x,t)\, q^m] = q\, h\, H_1(x,t) N[\phi(x,t,q)] \Rightarrow \\ L[\sum_{m=1}^{\infty} u_m(x,t)\, q^m] - q\, L[\sum_{m=1}^{\infty} u_m(x,t) q^m] = q\, h\, H_1(x,t) N[\phi(x,t,q)]\end{gathered} \tag{29}$$

By differentiating (29) m times with respect to q, we obtain

$$\{L[\sum_{m=1}^{\infty} u_m(x,t)\, q^m] - q\, L[\sum_{m=1}^{\infty} u_m(x,t) q^m]\}^{(m)} = \{q\, h\, H_1(x,t) N[\phi(x,t,q)]\}^{(m)} =$$

$$m!\, L[u_m(x,t) - u_{m-1}(x,t)] = h\, H_1(x,t)\, m \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}} \Big|_{q=0} .$$

Therefore,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t) \Re_m(u_{m-1}(x,t)), \tag{30}$$

where,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}} \Big|_{q=0}, \tag{31}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(30) is governing the linear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (31) for any nonlinear operator N.

To obtain the approximation solution of Eq.(1), according to HAM, let

$$N[u(x,t)] = u(x,t) - G(x,t) + \int_0^t F_1(u(x,t))\, dt + \int_0^t F_2(u(x,t))\, dt - \int_0^t F_3(u(x,t))\, dt,$$

so,

$$\Re_m(u_{m-1}(x,t)) = u_{m-1}(x,t) - G(x,t) + \int_0^t F_1(u_{m-1}(x,t))\,dt + \int_0^t F_2(u_{m-1}(x,t))\,dt$$
$$-\int_0^t F_3(u_{m-1}(x,t))\,dt. \tag{32}$$

Substituting (32) into (30)

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)[u_{m-1}(x,t) + \int_0^t F_1(u_{m-1}(x,t))\,dt$$
$$+\int_0^t F_2(u_{m-1}(x,t))dt - \int_0^t F_3(u_{m-1}(x,t))\,dt + (1-\chi_m)G(x,t)]. \tag{33}$$

We take an initial guess $u_0(x,t) = G(x,t)$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into (33) to give the recurrence relation

$$u_0(x,t) = G(x,t),$$
$$u_{n+1}(x,t) = -\int_0^t F_1(u_n(x,t))\,dt - \int_0^t F_2(u_n(x,t))\,dt + \int_0^t F_3(u_n(x,t))\,dt, \quad n \geq 0. \tag{34}$$

Therefore, the solution $u(x,t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= G(x,t) + \sum_{n=1}^{\infty}\left(-\int_0^t F_1(u_n(x,t))dt - \int_0^t F_2(u_n(x,t))\,dt + \int_0^t F_3(u_n(x,t))dt.\right). \tag{35}$$

Which is the method of successive approximations. If

$$|u_n(x,t)| < 1,$$

then the series solution (35) convergence uniformly.

2.4 Description of the HPM and MHPM

To explain HPM [30-36], we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u), \tag{36}$$

with initial conditions

$$u(x,0) = f(x).$$

According to HPM, we construct a homotopy which satisfies the following relation

$$H(u,p) = Lu - Lv_0 + p\,Lv_0 + p\,[Nu - f(u)] = 0, \tag{37}$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq.(37) is expressed as

$$u(x,t) = u_0(x,t) + p\,u_1(x,t) + p^2\,u_2(x,t) + ... \tag{38}$$

Hence the approximate solution of Eq.(36) can be expressed as a series of the power of p, i.e. $u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + ...$

where,

$$\begin{aligned} &u_0(x,t) = G(x,t),\\ &\vdots\\ &u_m(x,t) = \sum_{k=0}^{m-1} -\int_0^t F_1(u_{m-k-1}(x,t))\,dt - \int_0^t F_2(u_{m-k-1}(x,t))\,dt +\\ &\int_0^t F_3(u_{m-k-1}(x,t))\,dt, \quad m \geq 1. \end{aligned} \tag{39}$$

To explain MHPM [37-42], we consider Eq.(1) as

$$\begin{aligned} &L(u) = u(x,t) - G(x,t) + \int_0^t F_1(u(x,t))\,dt + \int_0^t F_2(u(x,t))\,dt\\ &-\int_0^t F_3(u(x,t))\,dt. \end{aligned}$$

Where $F_1(u(x,t)) = g_1(x)h_1(t)$, $F_2(u(x,t)) = g_2(x)h_2(t)$ and $F_3(u(x,t)) = g_3(x)h_3(t)$. We can define homotopy $H(u,p,m)$ by

$$H(u,0,m) = f(u), \quad H(u,1,m) = L(u),$$

where, m is an unknown real number and

$$f(u(x,t)) = u(x,t) - z(x,t).$$

Typically we may choose a convex homotopy by

$$H(u,p,m) = (1-p)\,f(u) + p\,L(u) + p\,(1-p)[m(g_1(x) + g_2(x) + g_3(x))] = 0, \quad 0 \leq p \leq 1. \tag{40}$$

Where m is called the accelerating parameters, and for $m = 0$ we define $H(u,p,0) = H(u,p),$ which is the standard HPM.

The convex homotopy (40) continuously trace an implicity defined curve from a starting point $H(u(x,t) - f(u),0,m)$ to a solution function $H(u(x,t),1,m)$. The embedding parameter p monotonically increase from 0 to 1 as trivial problem $f(u) = 0$ is continuously deformed to original problem $L(u) = 0$.

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \tag{41}$$

when $p \to 1$, Eq.(37) corresponds to the original one and Eq.(41) becomes the approximate solution of Eq.(1), i.e.,

$$u = \lim_{p \to 1} v = \sum_{m=0}^{\infty} u_m.$$

Where,

$$\begin{aligned}
&u_0(x,t) = G(x,t),\\
&u_1(x,t) = -\int_0^t F_1(u_0(x,t))\,dt - \int_0^t F_2(u_0(x,t))\,dt + \int_0^t F_3(u_0(x,t))\,dt - m(g_1(x)+g_2(x)+g_3(x)),\\
&u_2(x,t) = -\int_0^t F_1(u_1(x,t))\,dt - \int_0^t F_2(u_1(x,t))\,dt + \int_0^t F_3(u_1(x,t))\,dt + m(g_1(x)+g_2(x)+g_3(x)),\\
&\vdots\\
&u_m(x,t) = \sum_{k=0}^{m-1} -\int_0^t F_1(u_{m-k-1}(x,t))\,dt - \int_0^t F_2(u_{m-k-1}(x,t))\,dt + \int_0^t F_3(u_{m-k-1}(x,t))\,dt,\ m \geq 3.
\end{aligned} \tag{42}$$

3 Existence and convergency of iterative methods

We set,

$$\begin{aligned}
&\alpha_1 := T(L_1 + L_2 + L_3),\\
&\beta_1 := 1 - T(1-\alpha_1), \quad \gamma_1 := 1 - T\alpha_1.
\end{aligned}$$

Theorem 3.1 Let $0 < \alpha_1 < 1$, then Kadomtsev-Petviashvili equation (1), has a unique solution.

Proof. Let u and u^* be two different solutions of (3) then

$$\begin{aligned}
&|u-u^*| = |-\int_0^t [F_1(u(x,t)) - F_1(u^*(x,t))]\,dt - \int_0^t [F_2(u(x,t)) - F_2(u^*(x,t))]\,dt\\
&+\int_0^t [F_3(u(x,t)) - F_3(u^*(x,t))]\,dt| \leq \int_0^t |F_1(u(x,t)) - F_1(u^*(x,t))|\,dt + \int_0^t |F_2(u(x,t)) - F_2(u^*(x,t))|\,dt +\\
&\int_0^t |F_3(u(x,t)) - F_3(u^*(x,t))|\,dt \leq T(L_1+L_2+L_3)\,|u-u^*| = \alpha_1 |u-u^*|.
\end{aligned}$$

From which we get $(1-\alpha_1)|u-u^*| \leq 0$. Since $0 < \alpha_1 < 1$, then $|u-u^*| = 0$. Implies $u = u^*$ and completes the proof.

Theorem 3.2 The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem(1) using MADM convergence when

$$0 < \alpha_1 < 1, \ |u_1(x,t)| < \infty.$$

Proof. Denote as $(C[J],\|.\|)$ the Banach space of all continuous functions on J with the norm $\|G(x,t)\| = max\,|\,G(x,t)\,|$, for all x,t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\|s_n - s_m\| = max_{\forall t\in J}\ |\ s_n - s_m\ |= max_{\forall t\in J}\ |\ \sum_{i=m+1}^{n} u_i(x,t)\ |=$$

$$max_{\forall t\in J}\ |-\int_0^t (\sum_{i=m}^{n-1} A_i)dt - \int_0^t (\sum_{i=m}^{n-1} B_i)\ dt + \int_0^t (\sum_{i=m}^{n-1} L_i)\ dt\ |.$$

From [15], we have

$$\sum_{i=m}^{n-1} A_i = F_1(s_{n-1}) - F_1(s_{m-1}),$$
$$\sum_{i=m}^{n-1} B_i = F_2(s_{n-1}) - F_2(s_{m-1}),$$
$$\sum_{i=m}^{n-1} L_i = F_3(s_{n-1}) - F_3(s_{m-1}).$$

So,

$$\|s_n - s_m\| = max_{\forall t\in J}\ |-\int_0^t [F_1(s_{n-1}) - F_1(s_{m-1})]\,dt - \int_0^t [F_2(s_{n-1}) - F_2(s_{m-1})]\,dt + \int_0^t [F_3(s_{n-1}) - F_3(s_{m-1})]\,dt\,|\leq$$
$$\int_0^t |\,F_1(s_{n-1}) - F_1(s_{m-1})\,|\,dt + \int_0^t |\,F_2(s_{n-1}) - F_2(s_{m-1})\,|\,dt$$
$$+\int_0^t |\,F_3(s_{n-1}) - F_3(s_{m-1})\,|\,dt \leq \alpha_1 \|s_n - s_m\|.$$

Let $n = m+1$, then

$$\|s_n - s_m\| \leq \alpha_1 \|s_m - s_{m-1}\| \leq \alpha_1^2 \|s_{m-1} - s_{m-2}\| \leq \ldots \leq \alpha_1^m \|s_1 - s_0\|.$$

From the triangle inquality we have

$$\|s_n - s_m\| \leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \ldots + \|s_n - s_{n-1}\| \leq [\alpha_1^m + \alpha_1^{m+1} + \ldots + \alpha_1^{n-m-1}]\|s_1 - s_0\|$$
$$\leq \alpha_1^m [1 + \alpha_1 + \alpha_1^2 + \ldots + \alpha_1^{n-m-1}]\|s_1 - s_0\| \leq \alpha_1^m [\frac{1-\alpha_1^{n-m}}{1-\alpha_1}]\|u_1(x,t)\|.$$

Since $0 < \alpha_1 < 1$, we have $(1-\alpha_1^{n-m}) < 1$, then

$$\|s_n - s_m\| \leq \frac{\alpha_1^m}{1-\alpha_1} max_{\forall t\in J}\ |\,u_1(x,t)\,|. \tag{43}$$

But $|\,u_1(x,t)\,|< \infty$, so, as $m \to \infty$, then $\|s_n - s_m\| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete.

Theorem 3.3 The maximum absolute truncation error of the series solution

$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ to problem (1) by using MADM is estimated to be

$$max\,|u(x,t)-\sum_{i=0}^{m}u_i(x,t)|\le\frac{k\alpha_1^m}{1-\alpha_1}. \tag{44}$$

Proof. From inequality (43), when $n\to\infty$, then $s_n\to u$ and

$$max\,|u_1(x,t)|\le T(max_{\forall t\in J}\,|F_1(u_0(x,t))|+max_{\forall t\in J}\,|F_2(u_0(x,t))|+max_{\forall t\in J}\,|F_3(u_0(x,t))|).$$

Therefore,

$$\|u(x,t)-s_m\|\le\frac{\alpha_1^m}{1-\alpha_1}T(max_{\forall t\in J}\,|F_1(u_0(x,t))|+max_{\forall t\in J}\,|F_2(u_0(x,t))|+max_{\forall t\in J}\,|F_3(u_0(x,t))|).$$

Finally the maximum absolute truncation error in the interval J is obtained by (44).

Theorem 3.4 The solution $u_n(x,t)$ obtained from the relation (20) using VIM converges to the exact solution of the problem (1) when $0<\alpha_1<1$ and $0<\beta_1<1$.

Proof.

$$\begin{aligned}u_{n+1}(x,t)&=u_n(x,t)-L_t^{-1}([u_n(x,t)-G(x,t)+\int_0^t F_1(u_n(x,t))\,dt+\int_0^t F_2(u_n(x,t))\,dt\\&-\int_0^t F_3(u_n(x,t)))\,dt]),\end{aligned} \tag{45}$$

$$\begin{aligned}u(x,t)&=u(x,t)-L_t^{-1}([u(x,t)-G(x,t)+\int_0^t F_1(u(x,t))\,dt+\int_0^t F_2(u(x,t))\,dt\\&-\int_0^t F_3(u(x,t)))\,dt]).\end{aligned} \tag{46}$$

By subtracting relation (45) from (46),

$$\begin{aligned}&u_{n+1}(x,t)-u(x,t)=u_n(x,t)-u(x,t)-L_t^{-1}(u_n(x,t)-u(x,t)\\&+\int_0^t[F_1(u_n(x,t))-F_1(u(x,t))]\,dt+\int_0^t[F_2(u_n(x,t))-F_2(u(x,t))]\,dt-\\&\int_0^t[F_3(u_n(x,t))-F_3(u(x,t))]\,dt),\end{aligned}$$

if we set, $e_{n+1}(x,t)=u_{n+1}(x,t)-u_n(x,t)$, $e_n(x,t)=u_n(x,t)-u(x,t)$, $|e_n(x,t^*)|=max_t\,|e_n(x,t)|$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write,

$$
\begin{aligned}
&e_{n+1}(x,t) = e_n(x,t) + L_t^{-1}(-e_n(x,t) - \int_0^t [F_1(u_n(x,t)) - F_1(u(x,t))]\, dt \\
&- \int_0^t [F_2(u_n(x,t)) - F_2(u(x,t))]\, dt + r\int_0^t [F_3(u_n(x,t)) - F_3(u(x,t))]\, dt) \\
&\le e_n(x,t) + L_t^{-1}[-e_n(x,t) + L_t^{-1} \,|\, e_n(x,t) \,|\, (T(L_1 + L_2 + L_3)] \\
&\le e_n(x,t) - Te_n(x,\eta) + T(L_1 + L_2 + L_3) L_t^{-1} L_t^{-1} \,|\, e_n(x,t) \,| \\
&\le (1 - T(1-\alpha_1) \,|\, e_n(x,t^*) \,|,
\end{aligned}
$$

where $0 \le \eta \le t$. Hence, $e_{n+1}(x,t) \le \beta_1 \,|\, e_n(x,t^*) \,|$.
Therefore,

$$\|e_{n+1}\| = max_{\forall t \in J} \,|\, e_{n+1} \,| \le \beta_1 \; max_{\forall t \in J} \,|\, e_n \,| \le \beta_1 \|e_n\|.$$

Since $0 < \beta_1 < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete.

Theorem 3.5 The solution $u_n(x,t)$ obtained from the relation (22) using MVIM for the problem (1) converges when $0 < \alpha_1 < 1$, $0 < \gamma_1 < 1$.

Proof. The Proof is similar to the previous theorem.

Theorem 3.6 The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to problem (1) by using VIM is estimated to be

$$\|e_n\| \le \frac{\beta_1^n k'}{1-\beta_1}, \quad k' = max \,|\, u_1(x,t) \,|\,.$$

Proof.

$$
\begin{aligned}
&u_{n+1} - u_n = (u_{n+1} - u) + (u - u_n) = e_n - e_{n+1} \\
&\to e_n = e_{n+1} - (u_{n+1} - u_n) \\
&\|e_n\| = \|e_{n+1} - (u_{n+1} - u_n)\| \le \|e_{n+1}\| + \|u_{n+1} - u_n\| \le \beta_1 \|e_n\| + \|u_{n+1} - u_n\| \\
&\to \|e_n\| \le \frac{\|u_{n+1} - u_n\|}{1-\beta_1} \le \frac{\beta_1^n k'}{1-\beta_1}.
\end{aligned}
$$

Theorem 3.7 If the series solution (34) of problem (1) using HAM convergent then it converges to the exact solution of the problem (1).

Proof. We assume:

$$
\begin{aligned}
&u(x,t) = \sum_{m=0}^{\infty} u_m(x,t), \\
&\hat{F}_1(u(x,t)) = \sum_{m=0}^{\infty} F_1(u_m(x,t)), \\
&\hat{F}_2(u(x,t)) = \sum_{m=0}^{\infty} F_2(u_m(x,t)), \\
&\hat{F}_3(u(x,t)) = \sum_{m=0}^{\infty} F_3(u_m(x,t)).
\end{aligned}
$$

Where,

$$\lim_{m\to\infty} u_m(x,t) = 0.$$

We can write,

$$\sum_{m=1}^{n}[u_m(x,t)-\chi_m u_{m-1}(x,t)] = u_1 + (u_2 - u_1) + \ldots + (u_n - u_{n-1}) = u_n(x,t). \tag{47}$$

Hence, from (47),

$$\lim_{n\to\infty} u_n(x,t) = 0. \tag{48}$$

So, using (48) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = L[\sum_{m=1}^{\infty}[u_m(x,t)-\chi_m u_{m-1}(x,t)]] = 0.$$

therefore from (30), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = hH_1(x,t)\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0. \tag{49}$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation (49) and simplifying it , we have

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) &= \sum_{m=1}^{\infty}[\int_0^t F_1(u_{m-1}(x,t))\,dt \\ &+ \int_0^t F_2(u_{m-1}(x,t))\,dt - \int_0^t F_3(u_{m-1}(x,t))\,dt + (1-\chi_m)G(x,t)(x)] \\ &= u(x,t) - G(x,t) + \int_0^t \hat{F}_1(u(x,t))\,dt + \int_0^t \hat{F}_2(u(x,t))\,dt - \int_0^t \hat{F}_3(u(x,t))\,dt.\end{aligned} \tag{50}$$

From (49) and (50), we have

$$u(x,t) = G(x,t) - \int_0^t \hat{F}_1(u(x,t))\,dt - \int_0^t \hat{F}2(u(x,t))\,dt + \int_0^t \hat{F}3(u(x,t))\,dt.$$

Therefore, $u(x,t)$ must be the exact solution.

Theorem 3.8 The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to problem (1) by using HAM is estimated to be

$$\|e_n\| \le \frac{\alpha_1^n k^{'}}{1-\alpha_1}, \quad k^{'} = max\,|\,u_1(x,t)\,|.$$

Proof. The Proof is similar to the 3.6 theorem

Theorem 3.9 If $|\,u_m(x,t)\,|\le 1$, then the series solution $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ of problem (1) converges to the exact solution by using HPM.

Proof. We set,

$$\phi_n(x,t)=\sum_{i=1}^{n}u_i(x,t),$$
$$\phi_{n+1}(x,t)=\sum_{i=1}^{n+1}u_i(x,t).$$
$$|\,\phi_{n+1}(x,t)-\phi_n(x,t)\,|=D(\phi_{n+1}(x,t),\phi_n(x,t))=D(\phi_n+u_n,\phi_n)$$
$$=D(u_n,0)\le\sum_{k=0}^{m-1}\int_0^t|\,F_1(u_{m-k-1}(x,t))\,|\,dt+\int_0^t|\,F_2(u_{m-k-1}(x,t))\,|\,dt$$
$$+\int_0^t|\,F_3(u_{m-k-1}(x,t))\,|\,dt.$$
$$\rightarrow\sum_{n=0}^{\infty}\left|\phi_{n+1}(x,t)-\phi_n(x,t)\right|\le m\alpha_1\,|\,G(x,t)\,|\sum_{n=0}^{\infty}(m\alpha_1)^n.$$

Therefore,

$$\lim_{n\to\infty}u_n(x,t)=u(x,t).$$

Theorem 3.10 If $|\,u_m(x,t)\,|\le 1$, then the series solution $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ of problem (1) converges to the exact solution by using MHPM.

Proof. The Proof is similar to the previous theorem.

Theorem 3.11 The maximum absolute truncation error of the series solution $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ to problem (1) by using HPM is estimated to be

$$\|e_n\| \le \frac{(n\alpha_1)^n nk^{'}}{1-\alpha_1}, \quad k^{'} = max\,|\,u_1(x,t)\,|.$$

Proof. The Proof is similar to the 3.6 theorem

4 Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm 1:

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relations (10) for ADM , (13) for MADM, (34) for HAM, (39) for HPM and (42) for MHPM .

Step 3. If $|u_{n+1} - u_n| < \varepsilon$ then go to step 4,

else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Algorithm 2:

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relations (20) for VIM and (21) for MVIM.

Step 3. If $|u_{n+1} - u_n| < \varepsilon$ then go to step 4,

else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Example 4.1 *Consider the Kadomtsev-Petviashvili equation as follows:*

$$u_t(x,t) + (x+t)u_x(x,t) + \frac{1}{2}u_{xx}(x,t) - u(x,t) = 0.$$

With initial condition:

$$g(x) = e^x.$$

Table 1 Numerical results for Example 4.1

(x,t)	Errors			
	ADM(n=16)	MADM(n=13)	VIM(n=9)	MVIM(n=8)
(0.1,0.15)	0.081436	0.073123	0.050659	0.043416
(0.2,0.17)	0.082589	0.074356	0.051375	0.044237
(0.3,0.20)	0.083296	0.075419	0.051842	0.044732
(0.4,0.23)	0.083746	0.075729	0.052321	0.045144
(0.5,0.25)	0.084315	0.076348	0.052796	0.045748
(0.7,0.30)	0.085228	0.076808	0.053225	0.046207
(0.9,0.35)	0.086708	0.077173	0.053705	0.046875
(1.0,0.40)	0.087417	0.077783	0.054202	0.047089

(x,t)	Errors		
	HPM(n=9)	MHPM(n=8)	HAM(n=5)
(0.1,0.15)	0.062156	0.033646	0.022544
(0.20,0.17)	0.063015	0.034349	0.023154
(0.3,0.20)	0.063385	0.035172	0.023557
(0.4,0.23)	0.065317	0.035471	0.024048
(0.5,0.25)	0.065819	0.036163	0.024829
(0.7,0.30)	0.066112	0.036372	0.025362
(0.9,0.35)	0.066443	0.36724	0.026315
(1.0,0.40)	0.067236	0.037052	0.026759

Table 1, shows that, approximate solution of the Kadomtsev-Petviashvili equation is convergence with 5 iterations by using the HAM . By comparing the results of Table 1 , we can observe that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

5 Conclusions

The homotopy analysis method has been known as a powerful scheme for solving many functional equations such as algebraic equations, ordinary and partial differential equations, integral equations and so on. The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate the solution. In this work, the HAM has been successfully employed to obtain the approximate solution of the Kadomtsev-Petviashvili equation. We showed that the homotopy analysis method has more rapid convergence than the ADM, MADM, MVIM, HPM , MHPM and VIM.

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