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A New Goal Programming Approach for Cross Efficiency Evaluation

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Abstract Cross efficiency evaluation was developed as an extension of DEA. But the traditional DEA models usually have alternative optimal solutions and, as a result, cross efficiency scores may not be unique. It is recommended that without changing the DEA efficiency scores, the secondary goal should be introduced for optimization of the inputs/outputs weights. Several reports evaluated the performance ranking of DMUs by optimizing the rank position. These reports used ILP models for computation appropriate weights in cross efficiency evaluation. However, the LP models are easier and more applicable than ILP models. The present work proposes a goal programming model (LP model) that could be used as a secondary goal to choose suitable weights in cross efficiency evaluation. Also, the Numerical examples are provided to illustrate the approach.

Keywords: Data envelopment analysis (DEA); Cross efficiency; Goal programming; Ranking.

1 Introduction

Data envelopment analysis (DEA) permits us to measure the relative efficiency of a group of peer entities called decision-making units (DMUs) with common inputs/outputs. While DEA is an effective approach in efficiency evaluation, it might be criticized due to its flexibility in selection of inputs/outputs weights. Therefore, the cross evaluation method was developed as a DEA extension tool that can be utilized to identify best performing DMUs and to rank DMUs.

The main idea of cross efficiency evaluation is to use DEA in a peer evaluation mode instead of a self-evaluation mode, and there are two principal advantages: (1) it provides for a unique ordering of the DMUs; and (2) it eliminates unrealistic weight schemes without requiring the elicitation of weight restrictions from application area experts Anderson, Hollingsworth [1]. It is due to the cross efficiency evaluation has found a significant number of applications in various fields; see Chen [2], Wu et al. [3-5].

However, the DEA models usually have alternative optimal solutions and this non-uniqueness of the DEA optimal weights caused to arbitrarily generation of cross efficiency scores. It is due to this reason that the cross efficiency evaluation has also been extensively investigated theoretically. Sexton, Silkman [6] were the first who developed aggressive and

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benevolent formulations of cross efficiency to deal with the non-uniqueness issue. Doyle and Green [7, 8] presented slightly different secondary objective functions and showed how the cross efficiency evaluation could be used for various purposes. Similar thoughts also appeared in the articles of Anderson, Hollingsworth [1], Sun and Lu [9], Bao, Chen [10], Liang et al. [11, 12], Wu et al. [13, 14] , Lam [15], Jahanshahloo, Lotfi [16], Ramon et al [17, 18], Wang et al. [19-23], Lim [24], Soltanifar and Shahghobadi [25].

In the present paper, we improve the ILP approaches proposed in Wu, Liang [14] and Contreras [26], by providing an LP model from perspective of goal programming which optimizes the rank priority of DMU under evaluation. Because of solving LP models, the proposed method is applicable and much easier than related methods like Contreras [26]. The rest of the paper is organized as follows. Section 2 gives a brief introduction to the cross efficiency and its main formulations. The new model for cross efficiency evaluation is developed in section 3. Numerical examples are presented in section 4, while section 5 is devoted to concluding remarks.

2 Cross efficiency evaluation

Consider n DMUs, each of which consumes m inputs to produce s outputs. Denote by x_{ij} and y_{rj} the inputs/outputs values of DMU_j. The self-efficiency score for DMU_o is measured by CCR model as follows:

$$
\begin{aligned}
& Max \quad \theta_{oo} = \frac{\sum_{r=1}^{s} u_{ro} y_{ro}}{\sum_{i=1}^{m} v_{io} x_{io}} \\
& s.t. \\
& \qquad \frac{\sum_{r=1}^{s} u_{ro} y_{rj}}{\sum_{i=1}^{m} v_{io} x_{ij}} \le 1 \qquad j = 1,\ldots,n, \\
& \qquad u_{ro} \ge 0 \qquad\qquad r = 1,\ldots,s, \\
& \qquad v_{io} \ge 0 \qquad\qquad i = 1,\ldots,m.
\end{aligned} \tag{1}
$$

where θ^*_{oo} is called CCR-efficiency score of DMU_o and DMU_o is considered to be efficient if and only if $\theta^*_{oo} = 1$. Furthermore, model (1) can be transform to the following LP problem:

$$
\begin{aligned}
& Max \quad \theta_{oo} = \sum_{r=1}^{s} u_{ro} y_{ro} \\
& s.t. \\
& \qquad \sum_{i=1}^{m} v_{io} x_{io} = 1, \\
& \qquad \sum_{r=1}^{s} u_{ro} y_{rj} - \sum_{i=1}^{m} v_{io} x_{ij} \le 0, \qquad j = 1,\ldots,n, \\
& \qquad v_{io} \ge 0, \qquad\qquad i = 1,\ldots,m, \\
& \qquad u_{ro} \ge 0, \qquad\qquad r = 1,\ldots s.
\end{aligned} \tag{2}
$$

The self evaluation allows each DMU to be evaluated with the most favorable inputs/outputs weights so that θ^*_{oo} is the best relative efficiency score can be achieved for DMU_o, whereas the peer evaluation requests each DMU to be evaluated with the weights determined by the other DMUs [23]. In other words, peer evaluation of DMU_j using the most favorable weights of DMU_o is calculated based on the formula (3), where u* and v* are optimal solutions for model (2) when DMU_o is under evaluation:

$$\theta_{jo} = \frac{\sum_{r=1}^{s} u_{ro}^{*} y_{rj}}{\sum_{i=1}^{m} v_{io}^{*} x_{ij}}, \qquad j = 1, \ldots, n. \tag{3}$$

and finally

$$E_j = \frac{\sum_{k=1}^{n} \theta_{jk}}{n} \tag{4}$$

is referred as the cross efficiency score for DMU_j, which is simply the mean of the self and peer evaluations.

However, optimal weights obtained from model (2) are usually not unique. As a result, the efficiency scores defined in (3) are arbitrarily generated depending on optimal solution arising from the particular software in use [27]. Then, this non-uniqueness of the DEA optimal weights caused to arbitrarily generation of cross efficiency scores. To resolve this problem one remedy suggested by Sexton, Silkman [6] and was investigated by Doyle and Green (1994, 1995), is to introduce a secondary goal which optimizes the inputs/outputs weights while keeping unchanged the CCR efficiency scores.

Similar thoughts appeared in most of the theoretical papers about cross evaluation concept. Slightly different ideas can be found in Wu, Sun [28], Ruiz and Sirvent [29] and Yang, Ang [30]. Wu, Sun [28] proposed a weight balanced model where each DMU makes its own choice of weights without considering the effects on the other DMUs. Ruiz and Sirvent [29] make a choice of DEA weights looking for the profile without zeros with the least dissimilar weights, and then, they calculate the cross efficiency scores by using a weighted average of cross efficiencies in which the aggregation weights reflect the disequilibrium in the profiles of DEA weights used. Finally, Yang, Ang [30] consider all the possible weight sets in weight space, when cross efficiency scores are computed, and give each DMU an interval cross efficiency score.

3 Cross efficiency evaluation under the principle of rank priority of DMUs

To produce cross efficiency scores, Wu, Liang [14] and Contreras [26] were confining their attention to the case that the best ranking order is pursued for each DMU. Their ideas were based upon introducing ILP models to optimize the rank position of the DMU which is under consideration. However, solving ILP models are computationally intractable. Then, to improve their ILP approaches, we provide an LP model from perspective of goal programming which optimizes the rank priority of DMU under evaluation.

Liang, Wu [11] showed that determination of efficiency qualification can also be expressed equivalently based on the following deviation variable form:

$$
\begin{aligned}
&Min \quad s_o \\
&s.t. \\
&\qquad \sum_{i=1}^{m} v_{io} x_{io} = 1, \\
&\qquad \sum_{r=1}^{s} u_{ro} y_{rj} - \sum_{i=1}^{m} v_{io} x_{ij} + s_j = 0, && j = 1,\ldots,n, \\
&\qquad s_j \geq 0, && j = 1,\ldots,n, \\
&\qquad u_{ro} \geq 0, && r = 1,\ldots,s, \\
&\qquad v_{io} \geq 0, && i = 1,\ldots,m.
\end{aligned} \tag{5}
$$

where s_o is the deviation variable for DMU_o and s_j is the deviation variable for DMU_j (j=1,…,n). Based on this model, DMU_o is efficient if and only if $s_o^* = 0$.

Based on these deviation variables and by confining our attention to the case that the best ranking score is obtained for DMU_o, we need to minimize the slack variable s_o in such a way that it will be smaller than the other slack variables s_j (Note that $s_j \geq s_o \quad \forall j \in J$, implies that the efficiency score of DMU_o is larger than the DMUs in set J).

In the absence of such an ideal solution, a reasonable objective is to treat such inequalities as goal achievement in goal programming approach. Then, by definition of positive and negative deviation variables for such inequalities, we solve the following goal programming model to minimize sum of the negative deviation variables. In this manner, we develop following model in an effort to further prioritize the DMU_o against the other DMUs.

$$
\begin{aligned}
&Min \quad \sum_{j=1}^{n} \beta_j \\
&s.t. \\
&\qquad \sum_{r=1}^{s} u_{ro} y_{rj} - \sum_{i=1}^{m} v_{io} x_{ij} + s_j = 0, && j = 1,\ldots,n, \\
&\qquad \sum_{r=1}^{s} u_{ro} \bar{y}_{ro} - \sum_{i=1}^{m} v_{io} \bar{x}_{io} = 0, \\
&\qquad s_j - s_o - \alpha_j + \beta_j = 0, && j = 1,\ldots,n, \\
&\qquad u_{ro} \geq 0, && r = 1,\ldots,s, \\
&\qquad v_{io} \geq 0, && i = 1,\ldots,m, \\
&\qquad s_j \geq 0, \qquad \alpha_j \geq 0, \beta_j \geq 0 && j = 1,\ldots,n.
\end{aligned} \tag{6}
$$

where $(\bar{x}, \bar{y}) = (\theta_{oo}^* x_o, y_o)$ and θ_{oo}^* is the efficiency score of DMU_o, and α_j, β_j are positive and negative deviation variables for goal $s_j \geq s_o \quad \forall j \neq o$.

4 Numerical examples

In this section, we examine the validity of proposed cross efficiency model with three numerical examples and illustrate its potential applications.

Example 1: Sexton, Silkman [6] considered a case of six nursing homes whose inputs/outputs data for a given year are reported in Table 1.

Table 2 shows CCR efficiency scores in second column, where 4 of 6 units are efficient. The third column shows the cross efficiency scores produced based on the proposed GP model and fourth column shows cross efficiency scores produced by Doyle and Green [7] method. Moreover, Table 2 represents the rank of DMUs with three models in the parentheses, where results indicate that our ranking model in this case is close to model proposed by Doyle and Green [7] and in both models the most efficient unit is DMU_1.

Table 1 Inputs/outputs data

DMUs	Input1	Input2	Output1	Output2
DMU_1	1.5	0.2	14	35
DMU_2	4	0.7	14	210
DMU_3	3.2	1.2	42	105
DMU_4	5.2	2	28	420
DMU_5	3.5	1.2	19	250
DMU_6	3.2	0.7	14	150

Table 2 Efficiency scores and ranking

DMUs	CCR Eff.	GP Cross Eff.	Cross Eff. using Doyle model
DMU_1	1 (1)	1.1500 (1)	0.7639 (1)
DMU_2	1 (1)	0.9450 (3)	0.7004 (3)
DMU_3	1 (1)	0.9401 (4)	0.6428 (5)
DMU_4	1 (1)	1.0100 (2)	0.7184 (2)
DMU_5	0.9775 (5)	0.9394 (5)	0.6956 (4)
DMU_6	0.8675 (6)	0.9360 (6)	0.6081 (6)

Example 2: Table 3 shows the input/output data of seven academic departments in a university. Table 4 shows CCR and cross efficiency scores of them by different models where the rank of DMUs is depicted in parentheses. Second column listed the CCR efficiency scores where 6 of 7 DMUs are efficient and we can't distinguish them. In our new GP model and benevolent model, the most efficient DMU is DMU_6 and rank of DMUs in two models is close to each other.

Table 3 Inputs/outputs data

DMUs	Input1	Input2	Input3	Output1	Output2	Output3
DMU_1	12	400	20	60	35	17
DMU_2	19	750	70	139	41	40
DMU_3	42	1500	70	225	68	75
DMU_4	15	600	100	90	12	17
DMU_5	45	2000	250	253	145	130
DMU_6	19	730	50	132	45	45
DMU_7	41	2350	600	305	159	97

Table 4 Efficiency scores and ranking

DMU	CCR Eff.	Aggressive Cross Eff.	Benevolent Cross Eff.	GP Cross Eff.
DMU_1	1 (1)	0.8788 (1)	0.9442 (3)	0.9149 (2)
DMU_2	1 (1)	0.7219 (4)	0.9486 (2)	0.9030 (3)
DMU_3	1 (1)	0.7301 (3)	0.7827 (6)	0.7280 (7)
DMU_4	0.8197 (7)	0.4018 (7)	0.6160 (7)	0.8216 (6)
DMU_5	1 (1)	0.6259 (5)	0.8534 (5)	0.9000 (4)
DMU_6	1 (1)	0.8126 (2)	0.9801 (1)	0.9262 (1)
DMU_7	1 (1)	0.5966 (6)	0.8992 (4)	0.8870 (5)

With these two examples we showed that by using our new GP model, we can successfully compute cross efficiency scores and rank the DMUs which the results are close to Doyle and Green [7] results. But in example 3, we show that in some cases, the model suggested by Doyle and Green [7] can't explain the complete ranking.

Example 3: Cook and Kress [31] developed a DEA type model to rank the candidates in preferential election. The candidates are allowed to choose the most favorable weights to be applied to his/her standing. In this example, we utilized cross efficiency benevolent model and the proposed GP cross efficiency model. Suppose we have a case of twenty voters, each of whom is asked to rank four out of six candidates. The voting results are depicted in Table 5. For example, candidate b receives 4 first, 5 second, 5 third and 2 fourth placed votes.

Table 5 Votes achieved by candidates

Candidate	Standing1	Standing 2	Standing 3	Standing 4
a	3	3	4	3
b	4	5	5	2
c	6	2	3	2
d	6	2	2	6
e	0	4	3	4
f	1	4	3	3

Table 6 Efficiency scores and ranking

Candidate	CCR Eff.	Benevolent Cross Eff.	GP Cross Eff.
a	0.8125 (4)	0.800 (4)	0.8090 (4)
b	1 (1)	1.000 (1)	0.8330 (2)
c	1 (1)	0.922 (3)	0.7728 (6)
d	1 (1)	1.000 (1)	0.8350 (1)
e	0.6875 (5)	0.633 (6)	0.8250 (3)
f	0.6875 (5)	0.653 (5)	0.7960 (5)

Note that here with a same virtual input data, the additional weight restriction for outputs, used in Cook and Kress [31], is required because it means that the weight assigned to k^{th} place vote should be more than $(k+1)^{st}$ place vote. Hence, we have:

$$u_r - u_{r+1} \geq 0, \qquad r = 1,2,3 \tag{8}$$

The value of CCR efficiency scores for 6 candidates are listed in first column of Table 6 and 3 of 6 candidates are efficient. The second column of Table 6 shows cross efficiency of Doyle and Green [7] model where 2 of 6 candidates are efficient. In this case Doyle and Green [7] secondary goal can't give the complete ranking. However, the third column exhibits cross efficiency scores produced by proposed GP model and DMU_d has the best rank. In other words, candidate d is the best one and our model can explains the complete ranking.

5 Conclusion

Because DEA weights generally are not unique, the related cross efficiency evaluation may not be unique either. This non-uniqueness phenomenon can undermine the usefulness of the cross efficiency method. The present research investigates to improve ultimate cross efficiency score of DMUs that is achieved by introducing a new secondary objective function as a goal programming method. The proposed model optimizes the rank priority of the DMU under evaluation based on linear programing. The procedure is illustrated with different examples to show the potential of it.

References

1. Anderson, T. R., Hollingsworth, K., Inman, L., (2002). The fixed weighting nature of a cross-evaluation model, Journal of Productivity Analysis, 17(3), 249-255.
2. Chen, T. Y., (2002). An assessment of technical efficiency and cross-efficiency in Taiwan's electricity distribution sector, European Journal of Operational Research, 137(2), 421-433.
3. Wu, J., Liang, L., Chen, Y., (2009). DEA game cross-efficiency approach to Olympic rankings, Omega, 37(4), 909-918.
4. Wu, J., Liang, L., Yang, F., (2009). Achievement and benchmarking of countries at the Summer Olympics using cross efficiency evaluation method, European Journal of Operational Research, 197(2), 722-730.
5. Wu, J., Liang, L., Wu, D., Yang, F., (2008). Olympics ranking and benchmarking based on cross efficiency evaluation method and cluster analysis: the case of Sydney 2000, International Journal of Enterprise Network Management, 2(4), 377-392.
6. Sexton, T. R., Silkman, R. H., Hogan, A. J., (1986). Data envelopment analysis: Critique and extensions, New Directions for Evaluation, 1986(32), 73-105.
7. Doyle, J. Green, R., (1994). Efficiency and cross-efficiency in DEA: Derivations, meanings and uses, Journal of the operational research society, 45(5) 567-578.
8. Doyle, J. R., Green, R. H., (1995). Cross-Evaluation In Dea: Improving Discrimination Among Dmus, INFOR: Information Systems and Operational Research, 33(3), 205-222.
9. Sun, S., Lu, W. M., (2005). A cross-efficiency profiling for increasing discrimination in data envelopment analysis, INFOR: Information Systems and Operational Research, 43(1), 51-60.
10. Bao, C., Chen, T., Chang, S. Y., (2008). Slack-based ranking method: an interpretation to the cross-efficiency method in DEA, Journal of the Operational Research Society, 59(6), 860-862.
11. Liang, L., Wu, J., Cook, W. D., Zhu, J., (2008). Alternative secondary goals in DEA cross-efficiency evaluation, International Journal of Production Economics, 113(2), 1025-1030.
12. Liang, L., Wu, J., Cook, W. D., Zhu, J., (2008). The DEA game cross-efficiency model and its Nash equilibrium, Operations Research, 56(5), 1278-1288.
13. Wu, J., Liang, L., Yang, F., (2009). Determination of the weights for the ultimate cross efficiency using Shapley value in cooperative game, Expert Systems with Applications, 36(1), 872-876.
14. Wu, J., Liang, L., Zha, Y., Yang, F., (2009). Determination of cross-efficiency under the principle of rank priority in cross-evaluation, Expert Systems with Applications, 36(3), 4826-4829.
15. Lam, K., (2010). In the determination of weight sets to compute cross-efficiency ratios in DEA, Journal of the Operational Research Society, 61(1), 134-143.

16. Jahanshahloo, G., Lotfi, F. H., Jafari, Y., Maddahi, R., (2011). Selecting symmetric weights as a secondary goal in DEA cross-efficiency evaluation, Applied Mathematical Modelling, 35(1), 544-549.
17. Ramón, N., Ruiz, J. L., Sirvent, I., (2010). On the choice of weights profiles in cross-efficiency evaluations, European Journal of Operational Research, 207(3), 1564-1572.
18. Ramón, N., Ruiz, J. L., Sirvent, I., (2011). Reducing differences between profiles of weights: A "peer-restricted" cross-efficiency evaluation, Omega, 39(6), 634-641.
19. Wang, Y. M. Chin, K. S., (2010). A neutral DEA model for cross-efficiency evaluation and its extension, Expert Systems with Applications, 37(5), 3666-3675.
20. Wang, Y. M., Chin, K. S., (2010). Some alternative models for DEA cross-efficiency evaluation, International Journal of Production Economics, 128(1), 332-338.
21. Wang, Y. M., Chin, K. S., (2011). The use of OWA operator weights for cross-efficiency aggregation, Omega, 39(5), 493-503.
22. Wang, Y. M., Chin, K. S., Jiang, P., (2011). Weight determination in the cross-efficiency evaluation, Computers & Industrial Engineering, 61(3), 497-502.
23. Wang, Y.M., Chin, K. S., Luo, Y., (2011). Cross-efficiency evaluation based on ideal and anti-ideal decision making units, Expert systems with applications, 38(8), 10312-10319.
24. Lim, S., (2012). Minimax and maximin formulations of cross-efficiency in DEA, Computers & Industrial Engineering, 62(3), 726-731.
25. Soltanifar, M., Shahghobadi, S., (2013). Selecting a benevolent secondary goal model in data envelopment analysis cross-efficiency evaluation by a voting model, Socio-Economic Planning Sciences, 47(1), 65-74.
26. Contreras, I., (2012). Optimizing the rank position of the DMU as secondary goal in DEA cross-evaluation, Applied Mathematical Modelling, 36(6), 2642-2648.
27. Despotis, D. K., (2002). Improving the discriminating power of DEA: Focus on globally efficient units, Journal of the Operational Research Society, 53(3), 314-323.
28. Wu, J., Sun, J., Liang, L., (2012). Cross efficiency evaluation method based on weight-balanced data envelopment analysis model, Computers & Industrial Engineering, 63(2), 513-519.
29. Ruiz, J. L., Sirvent, I., (2012). On the DEA total weight flexibility and the aggregation in cross-efficiency evaluations, European Journal of Operational Research, 223(3), 732-738.
30. Yang, F., Ang, S., Xia, Q., Yang, C., (2012). Ranking DMUs by using interval DEA cross efficiency matrix with acceptability analysis, European Journal of Operational Research, 223(2), 483-488.
31. Cook, W. D., Kress, M., (1990). A data envelopment model for aggregating preference rankings, Management Science, 36(11), 1302-1310.